Outline

- Graph generalities
- Graph processing
- Storing the graph
- Aside: Variable-length representations
- Graph compression with instantaneous codes: BVGraph
- BVGraph for general graphs: LLPA
- Graph compression with Elias-Fano: EFGraph

Graph generalities

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- *Undirected graphs* can be safely identified with symmetric graphs.
- In an undirected graphs, nodes are often called *vertices* and pairs of opposite arcs are called *edges*.

Multigraphs and hypergraphs

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In some other applications, E is not a set of pairs, but a set of r-tuples. In this case, we talk of *hypergraphs*.

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A special case of labelling is the assingnment of real values, that is often called a *weighting function* (hence we call a graph node-weighted or arc-weighted).

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Typically, though, they are very *sparse*: A *sparse graph* is one with $O(n)$ arcs (instead of $O(n^2)$).

Paths

A *path* in G is a sequence $\pi = x_0, x_1, \ldots, x_k \in V$ such that $(x_i, x_{i+1}) \in E$ for all $i = 0, \ldots, k-1$. We say that:

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- if there is a cycle, G is called *cyclic*.

Strongly connected components

Let $x \approx y$ iff there is a path from x to y and vice-versa. The equivalence classes of $\approx$ are called the *strongly connected components* (SCCs) of G. The SCCs of G^s are called the *weakly connected components* (WCCs) of G: in the case of a symmetric graph, WCCs and SCCs coincide (and we just talk of "connected components").

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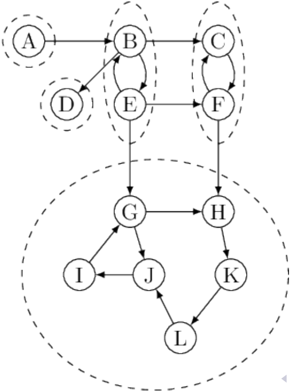

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The *reduced graph* $G^\dagger$ is the graph whose nodes are the SCCs of G, with an arc from $[x]$ to $[y]$ whenever there is a node $x' \approx x$ and a node $y' \approx y$ with $(x', y') \in E$.

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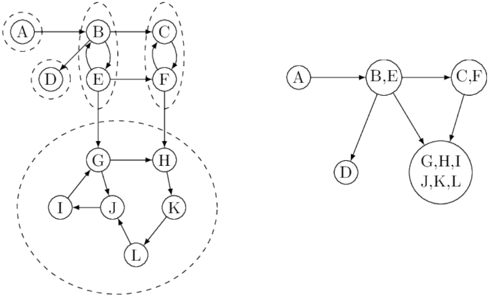

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- for undirected loopless graphs:

$$c_G(x) = \frac{2|E_G \cap (N_G(x) \times N_G(x)|}{d_G(x) \cdot (d_G(x) - 1)}$$

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A bijective graph morphism is called an *isomorphism*. If there exists an isomorphism between G and H we say that G and H are isomorphic, and write $G \cong H$.

Graph processing

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If we limit ourselves to isomorphism-invariant properties, we can assume w.l.o.g. that $V_G = \{0, 1, \dots, n-1\}$.

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  - if $G.E$ can be called only once (or $O(1)$ times), the access is "streaming".

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Typical problem:

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- for a given access mode
- write an algorithm that computes (or approximates, in some sense) $G \mapsto P(G)$

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Here: we only consider the former problem, not the latter!

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This means about $\log(n/d) + O(1)$ bits per arc. But *complex networks are NOT random graphs!*.

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- **highly unsuitable for sparse graphs!**

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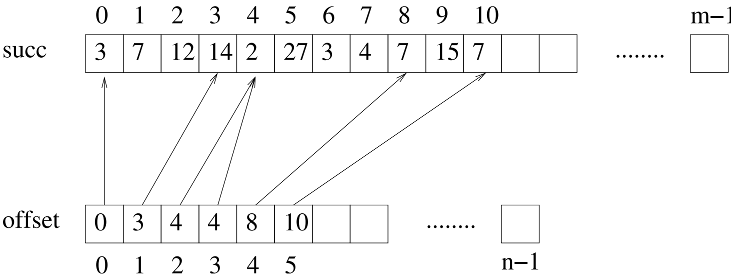
- scanning the graph sequentially takes time $O(m)$



The offset vector tells, for each given node x, where the successor list of x starts from. Implicitly, it also gives the degree of each node.

Naive representation

How much space does this representation take?

- Successor array: m elements (arcs), each containing a node ($\log n$ bits); with 32 bits, we can store up to 4 billion nodes (half of it, if we don't have unsigned types)

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All in all, $32(n + m)$ bits. If we assume $m = 8n$ (a very modest assumption on the outdegree), we need $288n$ bits, i.e., 288 bits/node, 36 bits/arc.
We show how to reduce this of an order of magnitude.

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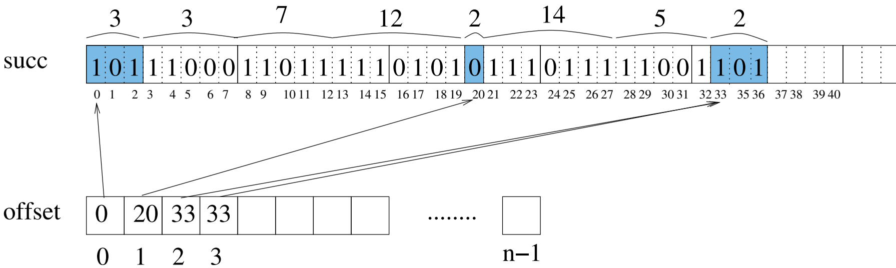
What about the offset array?

- bit displacement vs. byte displacement (with alignment)
- we have to keep an explicit representation of the node degrees (e.g., in the successor array, before every successsor list).

Variable-length representation



Node degrees (blue background), followed by successors. Each number is represented using an instantaneous code (possibly, different for degree and successors).

Aside: Variable-length representations

Instantaneous code

- An *instantaneous (binary) code* for the set S is a function $c : S \to \{0, 1\}^*$ such that, for all $x, y \in S$, if $c(x)$ is a prefix of $c(y)$, then $x = y$.

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- An instantaneous code for which the *equality* holds is called "complete".

Expected length

Given a set S and an instantaneous code $c : S \to \{0, \}^*$, the *expected length of c* with respect to some probability distribution $p : S \to [0, 1]$ is

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  - c_p^* is the Huffman coding.

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So, in practice, the choice of the code to use will be based on the expected distribution of the data.

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- Intended distribution:

$$p(x) = 2^{-\lceil \log N \rceil} \quad \text{uniform distribution.}$$

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0	1
1	01
2	001
3	0001
4	00001

A more general viewpoint

Unary coding can be seen as a special case of a more general kind of coding for **N**. Suppose you group **N** into *slots*: every slot is made by consecutive integers; let

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Then, to represent $x \in \mathbf{N}$ one can

- encode *in unary* the index i of the slot containing x;
- encode *in binary* the offset of x within its slot (using $\lceil \log s_i \rceil$ bits).

Golomb coding

Golomb coding with modulus b is obtained choosing

$$V = \langle b, b, b, \ldots \rangle.$$

To represent $x \in \mathbf{N}$ you need to specify the slot where x falls (that is, $\lfloor x/b \rfloor$) in unary, and then represent the offset using $\lceil \log b \rceil$ bits (or $\lfloor \log b \rfloor$ bits).

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So

$$l_x = \left\lfloor \frac{x}{b} \right\rfloor + \lceil \log b \rceil.$$

The intended distribution is

$$p(x) = 2^{-l_x} \propto (2^{1/b})^{-x} \quad \text{geometric distribution of ratio } 1/\sqrt[b]{2}.$$

More precisely...

A finer analysis shows that Golomb coding is optimal (=Huffman) for a geometric distribution of ratio p, provided that b is chosen as

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0	**1**0
1	**1**10
2	**1**11
3	**01**0
4	**01**10
5	**01**11
6	**001**0

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- Elias' γ is the first example we meet of a universal code!

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1	**1**
2	**010**0
3	**010**1
4	**011**00
5	**011**01
6	**00100**000
7	**00100**001

An alternative way. . .

. . . to think of γ coding is that x is represented using its usual binary representation (except for the initial "1", which is omitted), with every bit "coming with" a continuation bit, that tells whether the representation continues or whether it stops there.
For example (up to bit permutation) γ coding of 724 (in binary: 1011010100) is

0	1	1	1	1	1	0	1	1	1	0	1	1	1	0	1	0	0

k-bit-variable coding

What happens if we group digits k by k?

0 1 1 1 1 1 0 1 1 1 0 1 1 1 0 1 0 0

0 0 1 1 1 1 0 1 1 0 1 1 0 0 0

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k-bit-variable coding (cont'd)

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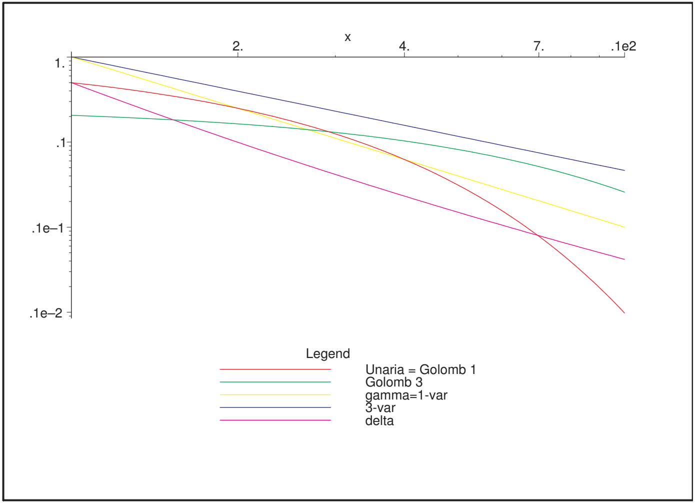
A more efficient variant: the ζ_k codes (for Zipf $1 \to 2$).

	$\gamma = \zeta_1$	ζ_2	ζ_3	ζ_4
1	1	10	100	1000
2	010	110	1010	10010
3	011	111	1011	10011
4	00100	01000	1100	10100
5	00101	01001	1101	10101
6	00110	01010	1110	10110
7	00111	01011	1111	10111
8	0001000	011000	0100000	11000

Comparing codings



Graph compression with instantaneous codes: BVGraph

Coding techniques. . .

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- Degrees are often distributed like a Zipf of exponent ≈ 2.7: they can be safely encoded using γ.
- What about successors? Let us assume that successors of x are $y_1, \ldots, y_k$: how should we encode $y_1, \ldots, y_k$?

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- Many hypertextual links contained in a web page are *navigational* ("home", "next", "up"...). If we compare the URL they refer to with that of the page containing them, they share a long common prefix. This property is known as *locality* and it was first observed by the authors of the Connectivity Server.

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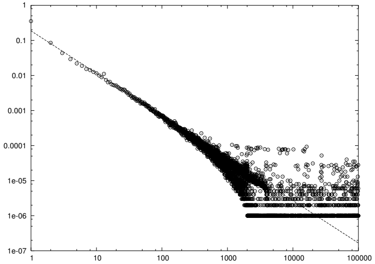
- To exploit this property, assume that URLs are ordered lexicographically (that is, node 0 is the first URL in lexicographic order, etc.). Then, if $x \to y$ is an arc, most of the times $|x - y|$ will be "small".

Exploiting locality

If x has successors $y_1 < y_2 < \cdots < y_k$, we represent its successor list though the gaps (*differentiation*):

$$y_1 - x, y_2 - y_1 - 1, \ldots, y_k - y_{k-1} - 1$$

(only the first value can be negative: wa make it into a natural number...). How are such differences distributed?



Zipf with exponent 1.2 $\implies$ we use ζ_3.

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We may encode the successor list of x as follows:

- we write the differences with respect to the successor list of some previous node $x - r$ (called the *reference node*)
- we explicitly encode (as before) only the successors of x that were not successors of $x - r$.

Similarity (cont'd)

More explicitly, the successor list of x is encoded as (*referencing*):

- an intger r (reference): if $r > 0$, the list is described by difference with respect to the successor list of $x - r$; in this case, we write a bitvector (of length equal to $d^+(x - r)$) discriminating the elements in $N^+(x - r) \cap N^+(x)$ from the ones in $N^+(x - r) \setminus N^+(x)$

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- an explicit list of *extra nodes*, containing the elements of $N^+(x) \setminus N^+(x - r)$ (or the whole $N^+(x)$, if $r = 0$), encoded as explained before.

Referencing example

Node	Outdegree	Successors
...	...	...
15	11	13, 15, 16, 17, 18, 19, 23, 24, 203, 315, 1034
16	10	15, 16, 17, 22, 23, 24, 315, 316, 317, 3041
17	0	
18	5	13, 15, 16, 17, 50
...	...	...

Node	Outd.	Ref.	Copy list	Extra nodes
...	...	...	...	...
15	11	0		13, 15, 16, 17, 18, 19, 23, 24, 203, 315, 1034
16	10	1	01110011010	22, 316, 317, 3041
17	0			
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Instead of using a bitvector, we use run-length encoding, telling the length of successive runs (blocks) of "0" and "1":

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Consecutivity

Among the extra nodes, many happen to sport the *consecutivity* property: they appear in clusters of consecutive integers. This phenomenon, observed empirically, have some possible explanations:

- most pages contain groups of navigational links that correspond to a certain hierarchical level of the website, and are often consecutive to one another;
- in the transpose graph, moreover, consecutivity is the dual of similarity with reference 1: when there is a cluster of consecutive pages with many similar links, in the transpose there are intervals of consecutive outgoing links.

Consecutivity (cont'd)

To exploit consecutivity, we use a special representation for the extra node list called *intervalization*, that is:

- sufficiently long (say $\geq T$) intervals of consecutive integers are represented by their left extreme and their length minus T;
- other extra nodes, if any, are called *residual nodes* and are represented alone.

Intervalization example

Node	Outd.	Ref.	# blocks	Copy blocks	Extra nodes
...	...	...	...	...	...
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Node	Outd.	Ref.	# bl.	Copy bl.s	# int.	Lft extr.	Lth	Residuals
...	...	...	...	...	...	...	...	...
15	11	0			2	15,...	4,...	13, 23 ...
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...	...	...	...	...	...	...	...	...

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The choice of W does not impact on decompression time.

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The choice of R influences the compression ratio (with $R = \infty$ giving the best possible compression) but also on decompression speed ($R = \infty$ may produce access time that can be two orders of magnitude larger than $R = 1$ — it may even produce stack overflows).

BVGraph for general graphs: LLPA

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From web graphs to complex networks

The basic property we have been exploiting so far is that *nodes are numbered according to the lexicographic ordering of URLs*. Is it possible to adapt / extend this idea to non-web graphs, e.g., to social networks?

- What we want is an ordering of the nodes that is compression friendly
- In particular, we want that most arcs are between nodes that are very close (as numbers) to each other.

Orderings and communities

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3. cluster sizes are going to be very unbalanced

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- ...with an adjustment depending on the overall popularity of the label

Label Propagation Algoritm (LPA)

Require: G a graph, γ a density parameter

$\pi \leftarrow$ a random permutation of G's nodes
for all x: $\lambda(x) \leftarrow x$, $v(x) \leftarrow 1$
while (some stopping criterion) **do**
    for $i = 0, 1, \ldots, n-1$ **do**
        for every label ℓ, $k_\ell \leftarrow |\lambda^{-1}(\ell) \cap N_G(\pi(i))|$
        $\hat{\ell} \leftarrow \text{argmax}_\ell[k_\ell - \gamma(v(\ell) - k_\ell)]$
        decrement $v(\lambda(\pi(i)))$
        $\lambda(\pi(i)) \leftarrow \hat{\ell}$
        increment $v(\lambda(\pi(i)))$
    end for
end while

Here $v(\ell)$ is the number of nodes currently labelled by ℓ, so $v(\ell) - k_\ell$ is the popularity of label ℓ outside of the current neighborhood.

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- Renumber nodes sorting by stably by the new labels

Name	LLP		BFS	Shingle		Natural		Random	
Amazon	9.16	(-30%)	12.96	14.43	(+11%)	16.92	(+30%)	23.62	(+82%)
DBLP	6.88	(-23%)	8.91	11.42	(+28%)	11.36	(+27%)	22.07	(+147%)
Enron	6.51	(-24%)	8.54	9.87	(+15%)	13.43	(+57%)	14.02	(+64%)
Hollywood	5.14	(-35%)	7.81	6.72	(-14%)	15.20	(+94%)	16.23	(+107%)
LiveJournal	10.90	(-28%)	15.1	15.77	(+4%)	14.35	(-5%)	23.50	(+55%)
Flickr	8.89	(-22%)	11.26	10.22	(-10%)	13.87	(+23%)	14.49	(+28%)
indochina (hosts)	5.53	(-17%)	6.63	7.16	(+7%)	9.26	(+39%)	10.59	(+59%)
uk (hosts)	6.26	(-18%)	7.62	8.12	(+6%)	10.81	(+41%)	15.58	(+104%)
eu	3.90	(-21%)	4.93	6.86	(+39%)	5.24	(+6%)	19.89	(+303%)
in	2.46	(-30%)	3.51	4.79	(+36%)	2.99	(-15%)	21.15	(+502%)
indochina	1.71	(-26%)	2.31	3.59	(+55%)	2.19	(-6%)	21.46	(+829%)
it	2.10	(-28%)	2.89	4.39	(+51%)	2.83	(-3%)	26.40	(+813%)
uk	1.91	(-33%)	2.84	4.09	(+44%)	2.75	(-4%)	27.55	(+870%)
altavista-nd	5.22	(-11%)	5.85	8.12	(+38%)	8.37	(+43%)	34.76	(+494%)

Graph compression with Elias-Fano: EFGraph

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- A very general technique!

Idea

Given a non-decreasing sequence:

$$0 \leq x_1, \ldots, x_d < n$$

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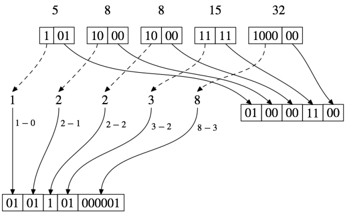

5, 8, 8, 15, 32 ≤ n = 36, ℓ = 2

Idea

So $x_1 = y_1 \cdot 2^\ell + r_1$, ..., $x_d = y_d \cdot 2^\ell + r_d$, with r_i written using ℓ bits each, and $g_1 = y_1 - 0, g_2 = y_2 - y_1, \ldots, g_d = y_d - y_{d-1}$ written in unary:

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The representation is almost optimal (Elias proves that it is $< .5$ bit away from the information-theoretic lower bound).