Link Analysis

Outline

- Ranking for information retrieval
- The web as a graph
- Centrality measures
- Two centrality measures: PageRank
- Two centrality measures: HITS

Ranking for information retrieval

Ranking, search engines, social networks

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Rankings may be composed (e.g., by linear combination): this is called *rank aggregation*.

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- *Selection*: it selects, from the set D of all available documents, a subset $S(q)$ of documents that satisfy q;
- *Ranking*: it establishes a total order on $S(q)$ determining how the results should be presented to the user.

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- ...now, the competitive edge is determined mostly by ranking!

Link Analysis — Problem and assumptions

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- **Basic assumption:** A link is a way to confer importance.

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This is called the *Web graph*.

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BE CAREFUL! Our knowledge of the Web is only indirect! (it is what we can collect using web crawlers, or interrogating search engines. . .)

What Is the Shape of the Web?

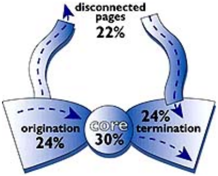
- The (known part of the) Web graph has a bowtie-like shape...
- Subdomains (e.g., country-code domains [`.fr`, `.it`, ...], large intranets etc.) are similar: a fractal-like graph
- It is *highly dynamic*

What Is the Shape of the Web?

Giant component (core)

- A strongly-connected component containing about 30% of the pages
- Estimated diameter: directed $\rightarrow$ 20/30; undirected $\rightarrow$ 10/17
- "It's a small-world!"

What Is the Shape of the Web?



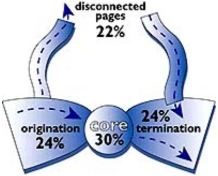
Left-hand side

- Ancestors of the giant component in the scc DAG
- About 25%
- Estimating its size is very difficult (how can you get there???)
- They are the Web pariahs: they are there, but no one wants to link them!

Right-hand side

- Descendants of the giant component in the scc DAG
- About 25%
- Contains all "documents" with no hyperlinks in them (e.g., pure text, word/PDF/PostScript with no links etc.)

What Is the Shape of the Web?



Tubes, tendrils and isolated components

- Remaining nodes (20%)

Ranking Techniques: A Taxonomy

Depending on whether the scoring (ranking) function depends or not on the query, and whether it depends or not on the text of the page (or only on its links):

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	Query-dependent (dynamic)	**Query-independent (static)**
Text-based	IR (already treated)	-
Link-based	e.g., HITS	e.g., PageRank

Centrality measures

Query-independent link-based techniques are methods that assign an *importance* to the nodes of a graph based on their position in the graph itself. This is often referred to as *graph centrality*

Centrality in networks

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Graph centrality is a topic of uttermost importance in social sciences. Briefly recalled in the next few slides.

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Freeman (1979) observed:

> *"several measures are often only vaguely related to the intuitive ideas they purport to index, and many are so complex that it is difficult or impossible to discover what, if anything, they are measuring"*

Types of centralities

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4. measures based on spectral measures [dominant eigenvector, Seeley's index, PageRank, HITS].

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- *harmonic* (Boldi, Vigna, 2013) $c_{\mathrm{harm}}(x) = \sum_{y \neq x} \frac{1}{d(y,x)}$

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- *Katz* (Katz, 1951): $c_{\text{Katz}}(x) = \sum_{t \geq 0} \beta^t p_t(x)$ where $p_t(x)$ is the number of paths of length t ending in x, and β is a parameter ($\beta < 1/\rho$)

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Two centrality measures: PageRank

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- it can be computed efficiently
- it is (used to be) the main ranking technique used at Google.

PageRank — An introductory metaphor (1)

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- At the beginning, everybody has the same amount of money.
- At every step, every page x gives away all of its money, redistributing it equally among its out-neighbors.

Problem with this solution: Formation of oligopolies that "suck away" all money from the system, without ever giving it back.

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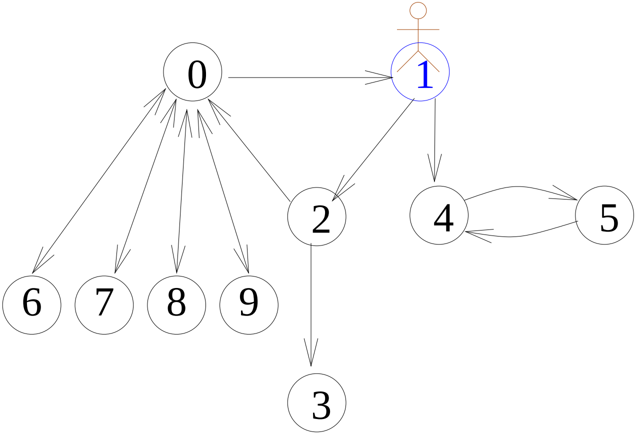
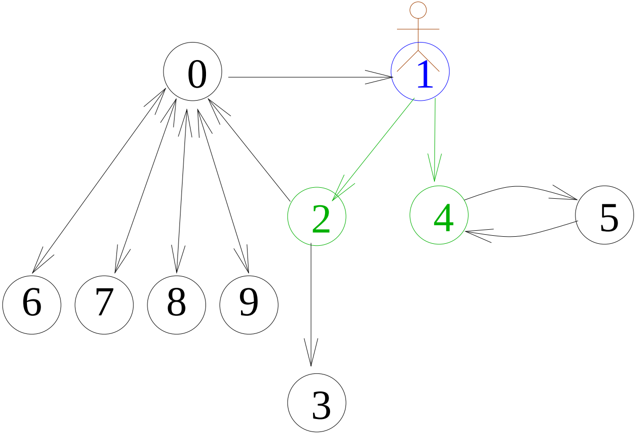
Dangling nodes pay, as every other node, $1 - \alpha$ in taxes, and distribute α to the nodes according to a fixed *dangling-node distribution* $\boldsymbol{u}$.

PageRank: the Web-Surfer Metaphor



A surfer is wandering about the web...



At each step, with probability α (s)he chooses the next page by clicking on a random link...

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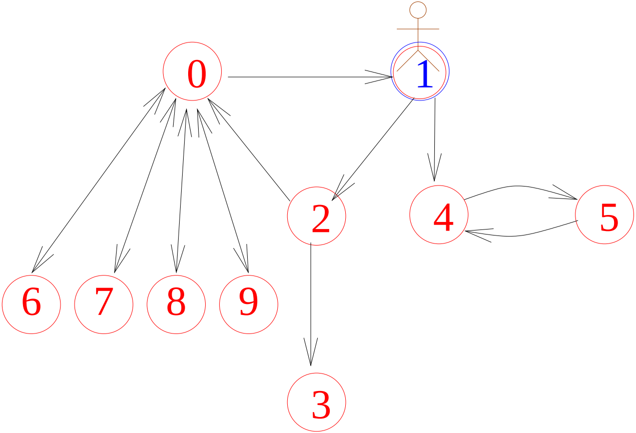

. . . with probability $1 - \alpha$, (s)he jumps to a random node (chosen uniformly or according to a fixed distribution, the *preference vector*)

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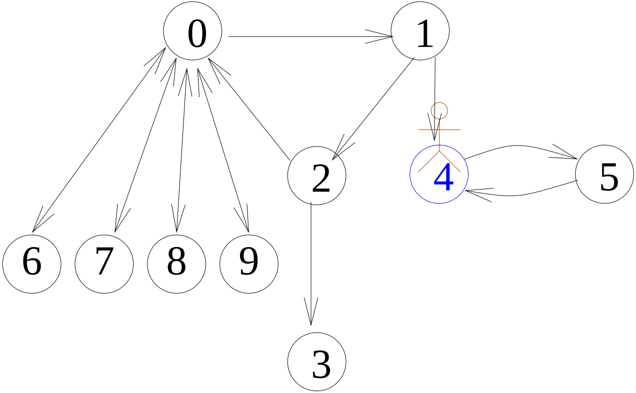

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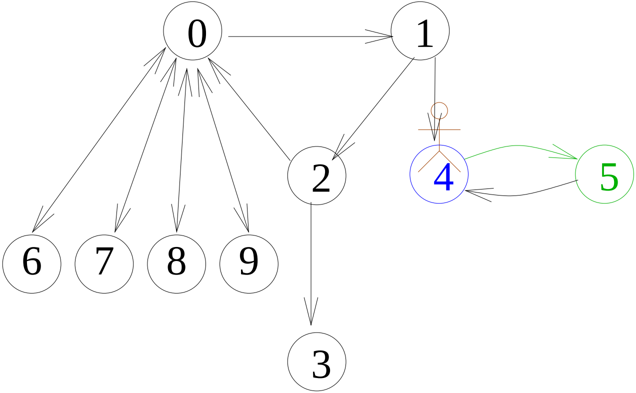

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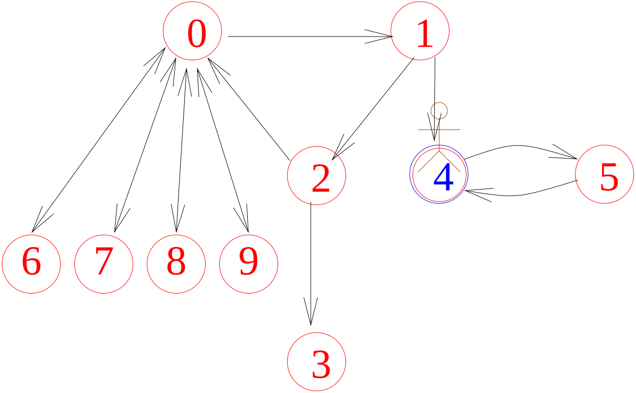

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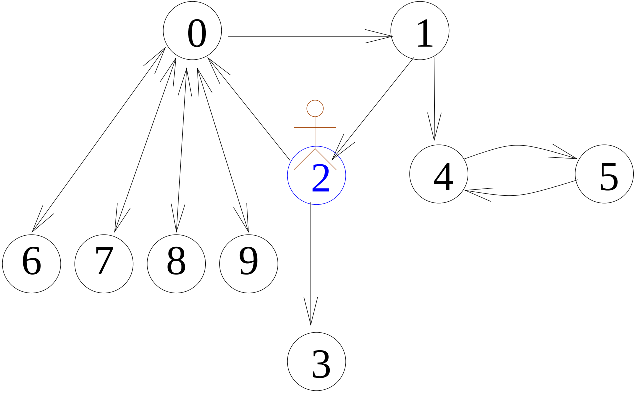

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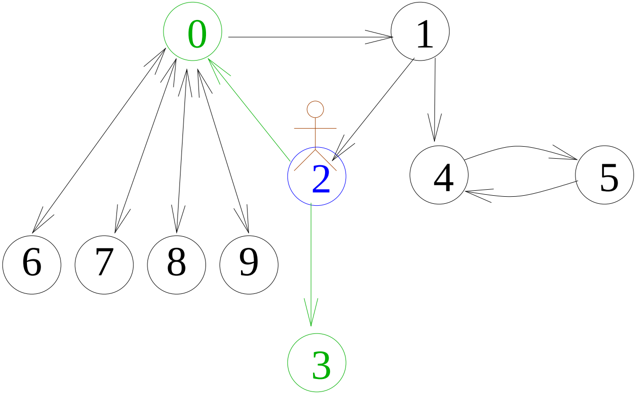

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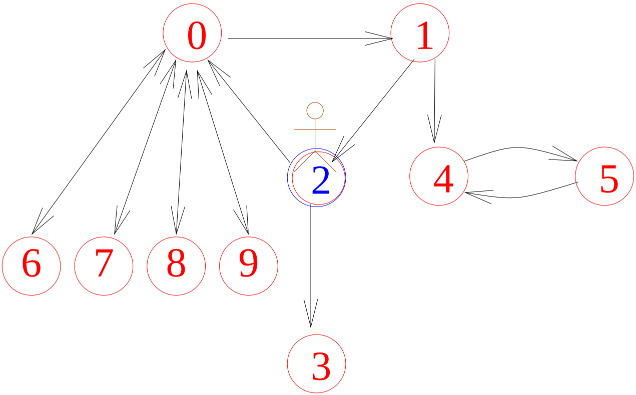

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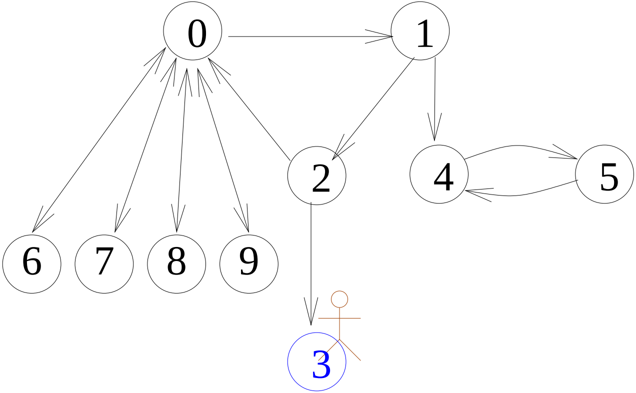

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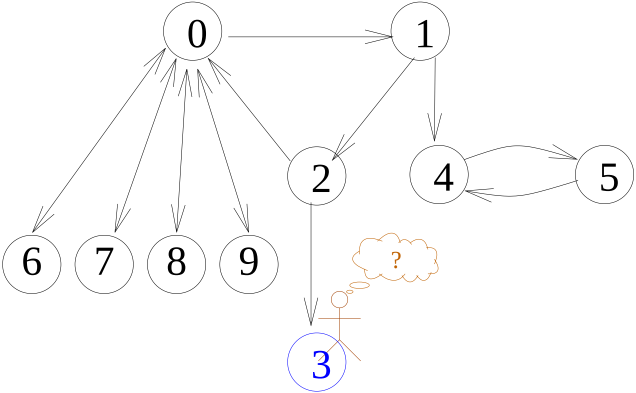

What if (s)he reaches a node with no outlinks (a *dangling node*)?

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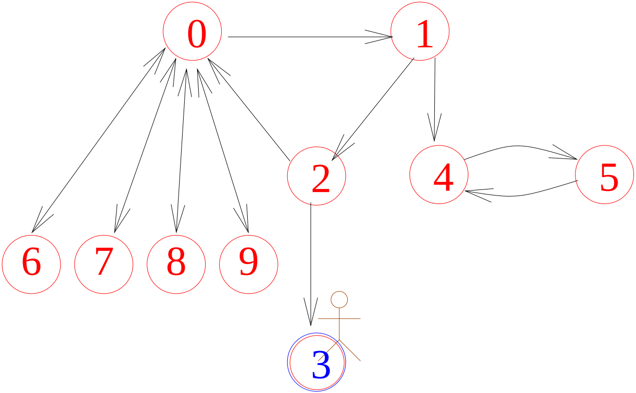

In that case, (s)he jumps to a random node *with probability 1*.

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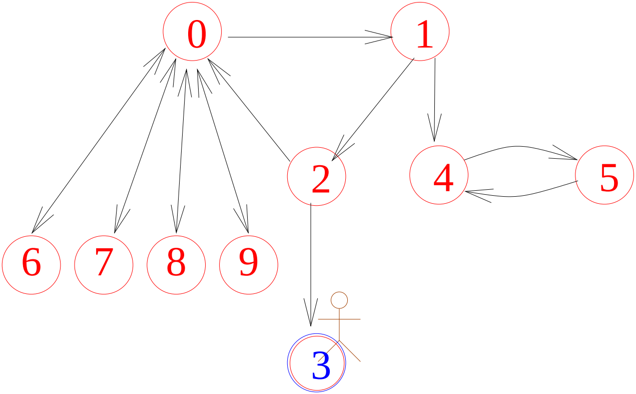

The PageRank of a page is the average fraction of time spent by the surfer on that page.

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How does PageRank depends on each of these factors? What happens at limit values (e.g., $\alpha \to 1$)?

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where $f(-)$ is a suitable *damping function* that goes to zero sufficiently fast [Baeza–Yates, Boldi & Castillo 2006].

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Theorem

The n-th approximation of PageRank computed by the Power Method with damping factor α and starting vector $\boldsymbol{v}$ coincides with the n-th degree Maclaurin polynomial of PageRank evaluated in α.

$$\boldsymbol{v}M^n = \boldsymbol{v} + \boldsymbol{v}\sum_{k=1}^{n}\alpha^k\left(P^k - P^{k-1}\right).$$

One α to rule them all. . .

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The difference between the k-th and the $(k-1)$-th approximation of PageRank (as computed by the Power Method with starting vector $\boldsymbol{v}$), divided by α^k, is the k-th coefficient of the power series of PageRank.

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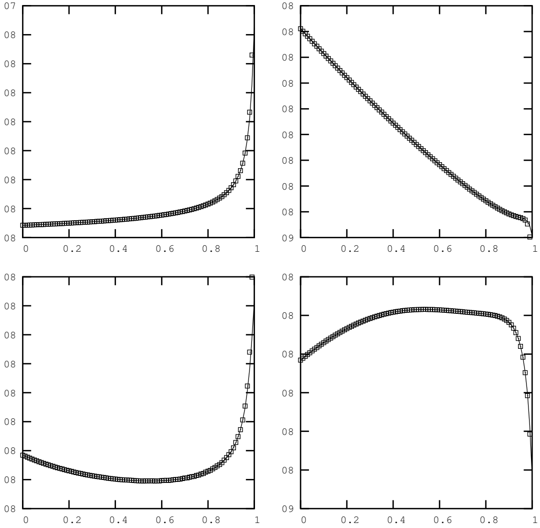
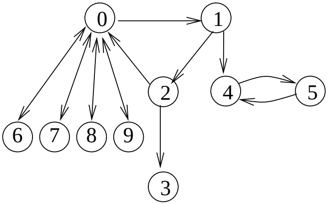
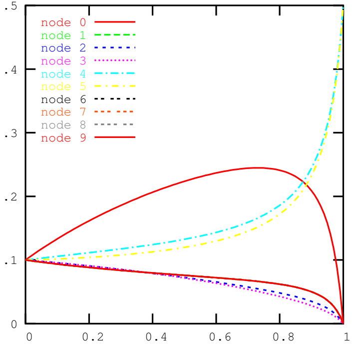
Even more is true, of course: using standard series derivation techniques, one can approximate the k-th derivative.

Some typical behaviours



An example





$$r_0(\alpha) = -5 \frac{(-1+\alpha)\left(\alpha^2 + 18\,\alpha + 4\right)}{8\,\alpha^4 + \alpha^3 - 170\,\alpha^2 - 20\,\alpha + 200}$$

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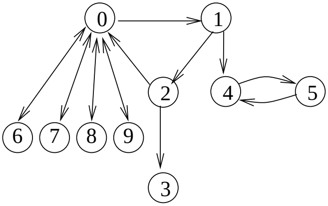

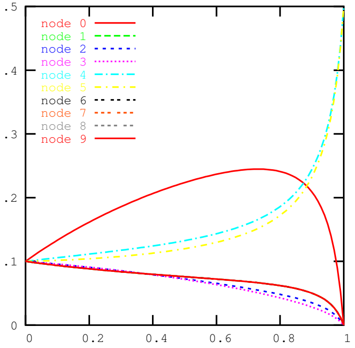

$$r_1(\alpha) = -2\,\frac{(-1+\alpha)\left(\alpha^2+2\,\alpha+10\right)}{8\,\alpha^4+\alpha^3-170\,\alpha^2-20\,\alpha+200}$$

The magic value $\alpha = 0.85$

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- ... numeric instability arises when α is too close to 1...
- ... yet, we believe that understanding how $\boldsymbol{r}(\alpha)$ changes when α is modified is important.

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Since $\boldsymbol{r}$ is a coordinatewise bounded function defined on $[0, 1)$, the limit

$$\boldsymbol{r}^* = \lim_{\alpha \to 1^-} \boldsymbol{r}$$

exists.

A ready-made solution

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In fact, since the *resolvent* $(I/\alpha - P)$ has a Laurent expansion around 1 in the largest disc not containing $1/\lambda$ for another eigenvalue λ of P, PageRank is analytic in the same disc; a standard computation yields

$$(1-\alpha)(1-\alpha P)^{-1} = P^* - \sum_{n=0}^{\infty}\left(\frac{\alpha-1}{\alpha}\right)^{n+1} Q^{n+1},$$

where $Q = (I - P + P^*)^{-1} - P^*$ and

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What makes $\boldsymbol{r}^*$ different from other limit distributions? How can we describe its structure?

Characterising r^*

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A node x of G is a *bucket* iff it is contained in a non-trivial strongly connected component with no arcs toward other components. (Non-trivial means that it contains at least one arc)

A characterisation theorem

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Corollary

Assume $\boldsymbol{u} = \mathbf{1}/n$. *Then:*

1. *if* G *contains a bucket then a node is recurrent for* P *iff it is a bucket;*
2. *if* G *does not contain a bucket all nodes are recurrent for* P.

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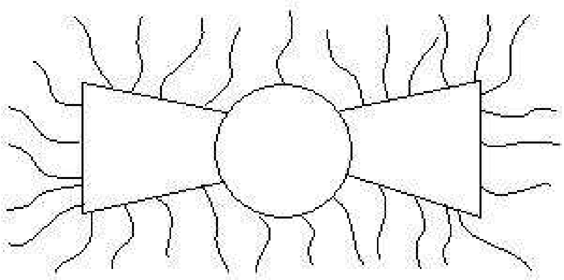
Theorem

1. *If a bucket of G is reachable from the support of $\boldsymbol{u}$ then a node is recurrent for P iff it is a bucket of G;*
2. *if no bucket of G is reachable from the support of $\boldsymbol{u}$, all nodes reachable from the support of $\boldsymbol{u}$ form a bucket component of P; hence, a node is recurrent for P iff it is in a bucket component of G or it is reachable from the support of $\boldsymbol{u}$.*

Bowtie

As a consequence, when $\alpha \to 1$, all PageRank concentrates in a bunch of pages that live in the rightmost part of the bowtie [Kumar et al., '00]:



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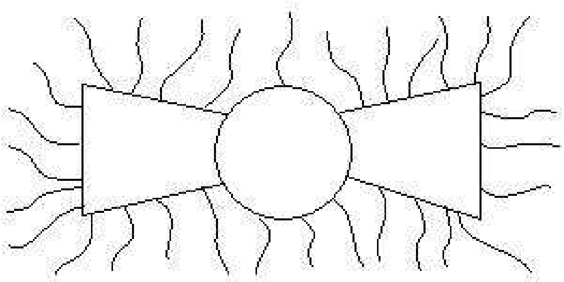

$\boldsymbol{r}(\alpha)$ becomes meaningless as $\alpha \to 1$!

Interpretation

The statement of the previous theorem may seem a bit unfathomable. The essence, however, could be stated as follows: except for strongly connected graphs, or graphs whose terminal components are dangling, **the recurrent nodes are exactly the buckets** (unless we are in the very pathological case in which no bucket is reachable from the support of $\boldsymbol{u}$).

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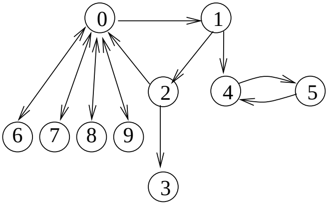
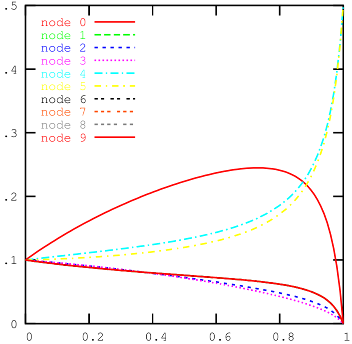
In a word: **PageRank when $\alpha \to 1$ is nonsense** in all real-world cases...

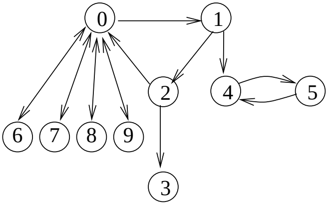
...and if you want the dire truth, there is an explicit formula in [Avrachenkov, Litvak & Kim 2006].

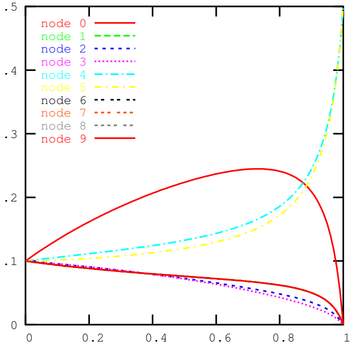
An example





$$r_0(\alpha) = -5\,\frac{(-1+\alpha)\left(\alpha^2 + 18\,\alpha + 4\right)}{8\,\alpha^4 + \alpha^3 - 170\,\alpha^2 - 20\,\alpha + 200}$$





$$r_1(\alpha) = -2\,\frac{(-1+\alpha)\left(\alpha^2 + 2\,\alpha + 10\right)}{8\,\alpha^4 + \alpha^3 - 170\,\alpha^2 - 20\,\alpha + 200}$$

General behaviour

What about the general behaviour of $\boldsymbol{r}$?

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Approximating them is also not difficult, since we have Maclaurin polynomials ($[\![\boldsymbol{r}^{(k)}(\alpha)]\!]_t$ is the polynomial of order t):

Theorem

If $t \geq k/(1-\alpha)$,

$$\big\|\boldsymbol{r}^{(k)}(\alpha) - [\![\boldsymbol{r}^{(k)}(\alpha)]\!]_t\big\| \leq \frac{\delta_t}{1-\delta_t}\big\|[\![\boldsymbol{r}^{(k)}(\alpha)]\!]_t - [\![\boldsymbol{r}^{(k)}(\alpha)]\!]_{t-1}\big\|,$$

where

$$1 > \delta_t = \frac{\alpha(t+1)}{t+1-k}.$$

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Also TotalRank is a special case of the general ranking technique of [Baeza–Yates, Boldi & Castillo 2006].

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Also TotalRank is a special case of the general ranking technique of [Baeza–Yates, Boldi & Castillo 2006]. The two damping functions for TotalRank and PageRank are:

$$\begin{aligned} d_T(\ell) &= \frac{1}{(t+1)(t+2)} \\ d_P(\ell) &= (1-\alpha)\alpha^{\ell}. \end{aligned}$$

. . . and a possible explanation for .85

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The average path length of the Web is about 20, and $\alpha^*(20) \approx .85\ldots$

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$$\boldsymbol{r} = (1-\alpha)\boldsymbol{v}(1-\alpha(\bar{G}+\boldsymbol{d}^T\boldsymbol{u}))^{-1}.$$

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- ...but the two versions are *very different!*: On a 100 million pages snapshot of the `.uk` domain, Kendall's τ is $\approx .25$ for a topic-based $\boldsymbol{v}$ and $\boldsymbol{u} = \mathbf{1}/n$! [Boldi *et al.* 2006]

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Using the Sherman–Morrison formula it is possible to make the dependence on $\boldsymbol{v}$ and $\boldsymbol{u}$ explicit, and sort out what happens in the strongly preferential case.

Pseudoranks

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Let us define the *pseudorank* of G with preference vector $\boldsymbol{v}$ and damping factor $\alpha \in [0\,..\,1]$:

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The notion appears in [Del Corso, Gullì & Romani 2004] and it has been used in [McSherry 2005; Fogaras, Rácz, Csalogány & Sarlós 2005] (actually, as *the* definition of PageRank).

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Using pseudoranks we can easily express the dependence [Boldi, Posenato, Santini & Vigna 2006]:

$$\boldsymbol{r} = \widetilde{\boldsymbol{v}}(\alpha) - \frac{\widetilde{\boldsymbol{v}}(\alpha)\boldsymbol{d}^T}{1 - \frac{1}{\alpha} + \widetilde{\boldsymbol{u}}(\alpha)\boldsymbol{d}^T}\widetilde{\boldsymbol{u}}(\alpha).$$

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Another evident feature of the above formula is that the dependence on the dangling-node distribution is *not linear*, so *we cannot expect strongly preferential PageRank to be linear in* $\boldsymbol{v}$.

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Nonetheless, if we fix $\boldsymbol{u} = \boldsymbol{v}$ and simplify the resulting formula (getting back the formula obtained by Del Corso, Gullì and Romani)...

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Assuming that $\boldsymbol{v} = \lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}$, we have

$$\boldsymbol{r} = \boldsymbol{r}_{\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}}(\alpha) \quad \propto \quad \lambda\widetilde{\boldsymbol{x}}(\alpha) + (1-\lambda)\widetilde{\boldsymbol{y}}(\alpha)$$

Two centrality measures: HITS

Alternatives to PageRank

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- HITS (Kleinberg)
- SALSA (Lempel, Moran), a variant of HITS (not covered here)

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Differently from PageRank it is *not query independent*.

HITS (cont'd)

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- a graph G_q (a subgraph of the whole web graph) is singled out (depending on the query)
- the authoritativeness/hubbiness scores are computed for the pages in G_q

HITS — Phase 1

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G_q is obtained as follows:

- the set S_q of the top k pages relative to q are obtained using some techniques (e.g., BM25)
- for each $x \in S_q$, all nodes in $N^+(x)$ are added
- for each $x \in S_q$, at most h nodes of $N^-(x)$ are added

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At every iteration, we will have two scores $h_x(t)$ and $a_x(t)$ for every node $x \in N_{G_q}$.

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\begin{aligned}
h_x(t+1) &\propto \sum_{x \rightarrow y} a_y(t) \\
a_x(t+1) &\propto \sum_{x \leftarrow y} h_y(t)
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h_x(t+1) &\propto \sum_{x \to y} a_y(t) \\
a_x(t+1) &\propto \sum_{x \leftarrow y} h_y(t)
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The $\propto$ is necessary to avoid divergence (the scores are normalized at every iteration).

HITS — In practice

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- its dynamic nature (requiring computation at query time)
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It was supposedly used by Teoma (later Ask.com).

Axioms for centrality

Back to centrality measures

A *centrality measure* (or index) is a way to assign a score to every node of a graph that represents the importance of that node (the larger the score, the more important the node).

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PageRank and HITS are but two examples (both are *spectral* because they are related to the spectrum of some suitable matrix derived from the graph).

Assessing

Is there a robust way to convince oneself that a certain centrality measure is better than another?

Axiomatization. . .

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- ...hard axioms (characterize a centrality measure completely)
- ...soft axioms (like the T_i axioms for topological spaces)

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- if $p \to \infty$, every node of the cycle becomes ultimately strictly more important than every node of the clique

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- if $k \to \infty$, the node on the clique-side of the bridge becomes more important than the node on the cycle-side.

Score monotonicity

Adding an arc $x \to y$ strictly increases the score of y.

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Doesn't say anything about the score of other nodes!

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- if y used to dominate z, then the same holds after adding the arc
- if y had the same score as z, then the same holds after adding the arc
- **strict variant:** if y had the same score as z, then y dominates z after adding the arc

Rank monotonicity

	Monotonicity				Other axioms	
	General		Strongly connected			
Centrality	Score	Rank	Score	Rank	Size	Density
Harmonic	yes	yes*	yes	yes*	yes	yes
Degree	yes	yes*	yes	yes*	only k	yes
Katz	yes	yes*	yes	yes*	only k	yes
PageRank	yes	yes*	yes	yes*	no	yes
Dominant	no	no	yes	yes*	only k	yes
Seeley	no	no	yes	yes	no	yes
Lin	no	no	yes	yes	only k	no
Closeness	no	no	yes	yes	no	no
HITS	no	no	no	no	only k	yes
SALSA	no	no	no	no	no	yes
Betweenness	no	no	no	no	only p	no