Teorema: Se $P \neq NP$, non esiste un algoritmo polinomiale α-approssimante per CENTER SEL con $\alpha < 2$.

Dim:

DOMINATING SET

INPUT: $G=(V,E)$, $k \in \mathbb{N}^{>0}$

OUTPUT: $\exists D \subseteq V$ t.c.

$|D| \leq k$ t.c.

$\forall x \in V \setminus D \quad \exists y \in D$

$xy \in E$

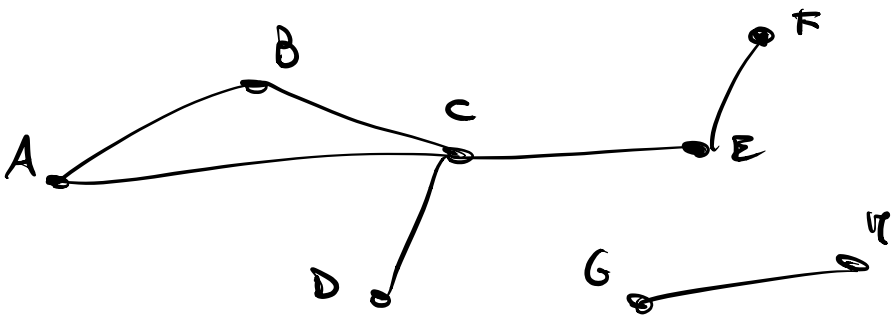
Lemma: DOMINATING SET $\in$ NP-Compl.

INPUT per DOMINATING SET: $G=(V,E)$, k

" " CENTER SELECTION: S, k

$d: S \times S \to \mathbb{R}$

$$d(x,y) = \begin{cases} 0 & \text{se } x = y \\ 1 & \text{se } x \neq y,\ xy \in E \\ 2 & \text{se } x \neq y,\ xy \notin E \end{cases}$$



	A	B	C	D	E	F	G	H
A	0	1	1	2	2	2	2	2
B		0	1	2	2	2	2	2
C			0	1	1	2	2	2
D								
E								
F								
G								
H								

$$\underbrace{d(x,y)}_{1 \text{ o } 2} \leq \underbrace{d(x,z) + d(z,y)}_{2 \text{ o } 3 \text{ o } 4}$$

$$\rho^*(S,k) \in \{1,2\}$$

$\rho^*(S,k) = 1$

<u>sse</u> $\exists\, C \subseteq S \quad |C| \le k$

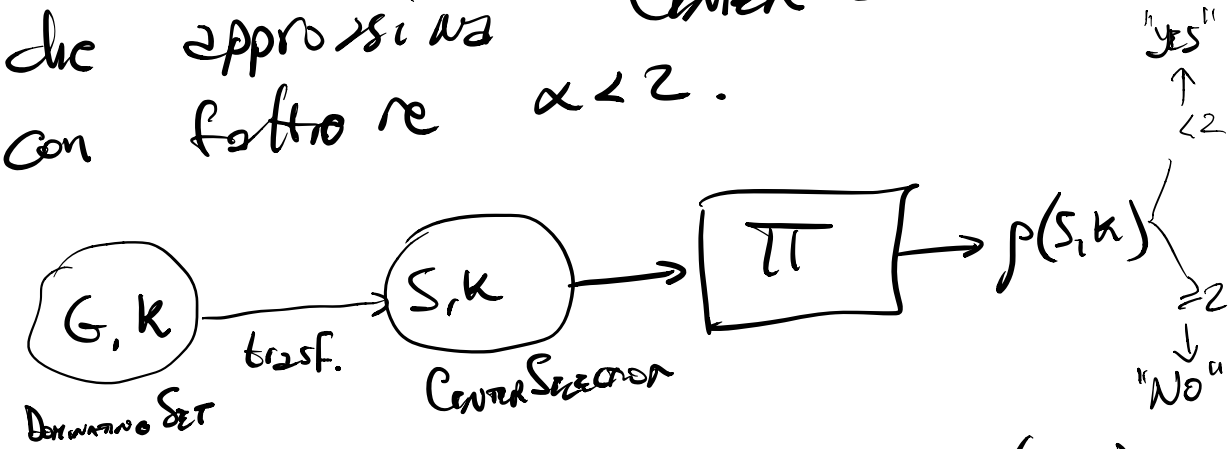
t.c. $\forall x \in S \setminus C \quad \exists c \in C$

$d(x,c) = 1$

<u>sse</u> $xc \in E$

<u>sse</u> la risposta per dominating set è "YES"

Per assurdo, supponiamo che esista un algoritmo Π in tempo pol. che approssima CENTER SELECTION con fattore $\alpha < 2$.



$$\rho^*(S,k) \le \rho(S,k) \le \alpha\, \rho^*(S,k)$$

$$\rho^*(S,k) \le \rho(S,k) < 2\rho^*(S,k)$$

$\rho^* = 1$

$\rho^* = 2$

$$1 \le \rho(S,k) < 2$$

$$2 \le \rho(S,k) < 4$$

$\square$

(MINIMUM) SET COVER

FUNZIONE ARMONICA

$$H : \mathbb{N}^{>0} \to \mathbb{R}$$

$$H(n) = \sum_{i=1}^{n} \frac{1}{i}$$

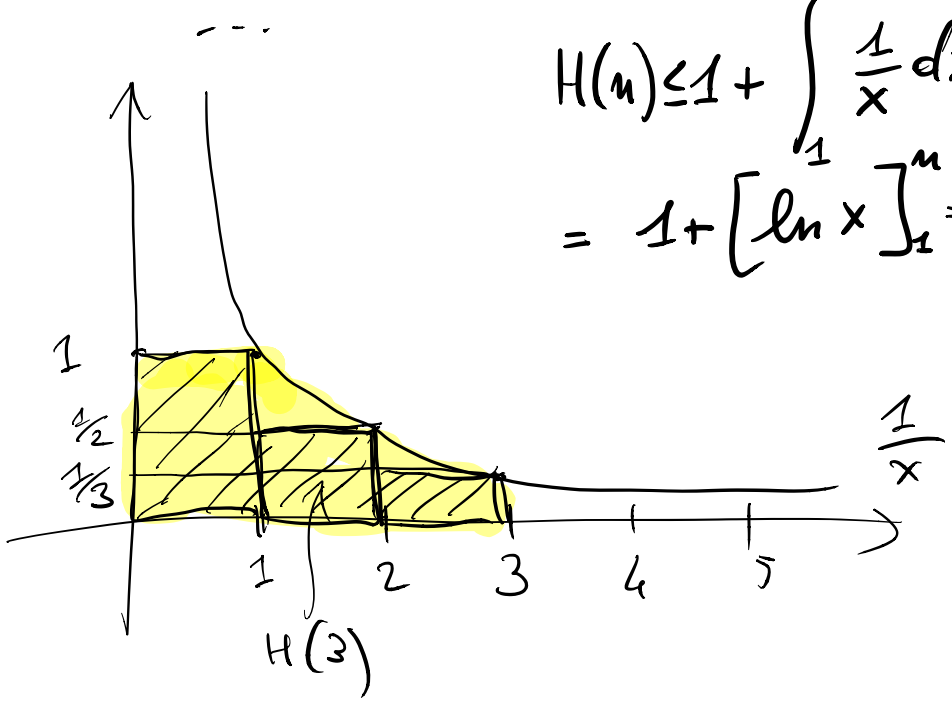
$$H(1) = 1$$

$$H(2) = 1 + \frac{1}{2}$$

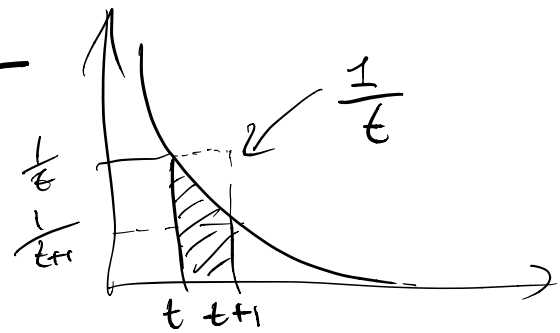
$$H(3) = 1 + \frac{1}{2} + \frac{1}{3}$$

$$H(n) \leq 1 + \int_1^n \frac{1}{x}\,dx =$$

$$= 1 + \left[\ln x\right]_1^n = 1 + \ln n$$



$$\int_t^{t+1} \frac{1}{x}\,dx \le \frac{1}{t}$$



$$H(n) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \ge$$

$$\ge \int_1^2 \frac{1}{x}\,dx + \int_2^3 \frac{1}{x}\,dx + \dots + \int_n^{n+1} \frac{1}{x}\,dx =$$

$$= \int_1^{n+1} \frac{1}{x}\,dx = \Big[\ln x\Big]_1^{n+1} =$$

$$= \ln(n+1)$$

$$\boxed{\ln(n+1) \le H(n) \le 1 + \ln n}$$

PROBLEMA SET COVER

INPUT: $S_1, S_2, \dots, S_m$ $\quad \bigcup_{i=1}^{m} S_i = U$

$|U| = n$

$w_1, w_2, \dots, w_m \in \mathbb{Q}^{\geq 0}$

SOLUZIONE AMMISSIBILE:

$\mathcal{C} \subseteq \{S_1, \dots, S_m\}$

t.c. $\bigcup_{S_i \in \mathcal{C}} S_i = U$

OBIETTIVO: $W = \sum_{S_i \in \mathcal{C}} w_i$

TIPO = MIN

GREEDY SET COVER

$R \leftarrow U$

$\mathcal{C} \leftarrow \emptyset$

while $R \neq \emptyset$

- scegli S_i che minimizzi $\dfrac{w_i}{|S_i \cap R|}$
- $\mathcal{C} \leftarrow \mathcal{C} \cup \{S_i\}$
- $R \leftarrow R \setminus S_i$

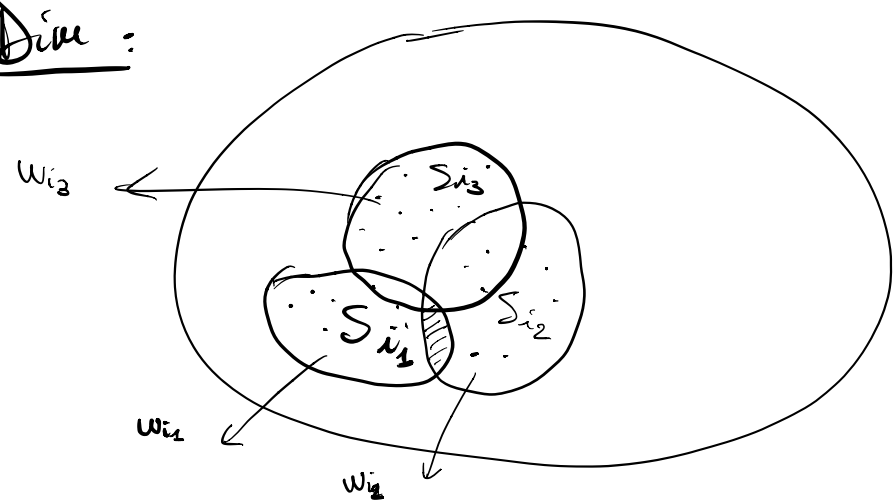
output $\mathcal{C}$

$\forall s \in S_i \cap R$

$c_s \triangleq \dfrac{w_i}{|S_i \cap R|}$

Lemma 1: $w = \sum_{s \in U} c_s$

Dim:



Lemma 2: Per ogni k

$$\sum_{s \in S_k} c_s \leq H(|S_k|) \cdot w_k$$

Dim: $S_k = \{s_1, s_2, \dots, s_d\}$

numerati in ordine di copertura

Vale per ogni $j = 1, \dots, d$

Quando viene coperto s_j mediante S_h. In quel momento

$$R \supseteq \{s_j, s_{j+1}, \dots, s_d\}$$

Quindi

$$|S_k \cap R| \geq d - j + 1$$

Allora

$$c_{s_j} = \frac{w_h}{|S_h \cap R|} \leq \frac{w_k}{|S_k \cap R|} \leq \frac{w_k}{d-j+1}$$

perché h è scelto in modo da minimizzare

$$\sum_{s \in S_k} c_s = c_{s_1} + c_{s_2} + \dots + c_{s_d} \leq$$

$$\leq \sum_{j=1}^{d} \frac{w_k}{d-j+1} =$$

$$= \frac{w_k}{d} + \frac{w_k}{d-1} + \dots + \frac{w_k}{2} + \frac{w_k}{1} =$$

$$= w_k \left(1 + \frac{1}{2} + \dots + \frac{1}{d}\right) =$$

$$= H(d)\, w_k =$$

$$= H(|S_k|)\, w_k . \qquad \square$$

<u>Teorema</u>: GREEDY SET COVER fornisce una $H(M)$-approssimazione per SET COVER, dove

$$M = \max_i |S_i| .$$

<u>Dim</u>: Peso ottimo $w^* = \sum_{S_i \in \mathcal{C}^*} w_i$.

Lemma 2

$$(*) \quad w_i \geq \frac{\sum_{s \in S_i} c_s}{H(|S_i|)} \geq \frac{\sum_{s \in S_i} c_s}{H(M)}$$

Siccome gli $S_i \in \mathcal{C}^*$ sono una copertura

$$(**) \quad \sum_{S_i \in \mathcal{C}^*} \sum_{s \in S_i} c_s \geq \sum_{s \in U} c_s \underset{\text{Lemma 1}}{=} w .$$

$$w^* = \sum_{S_i \in \mathcal{C}^*} w_i \underset{(*)}{\geq} \sum_{S_i \in \mathcal{C}^*} \frac{\sum_{s \in S_i} c_s}{H(M)} \underset{(**)}{\geq} \frac{w}{H(M)}$$

$$\Rightarrow \frac{w}{w^*} \leq H(M). \qquad \square$$

$$M \leq |U| = n$$

$$\Rightarrow H(M) \leq H(n) \leq 1 + \ln n$$

<u>Corollario</u>: Asintoticamente, fornisce GREEDY SET COVER una $O(\log n)$-approssimazione.

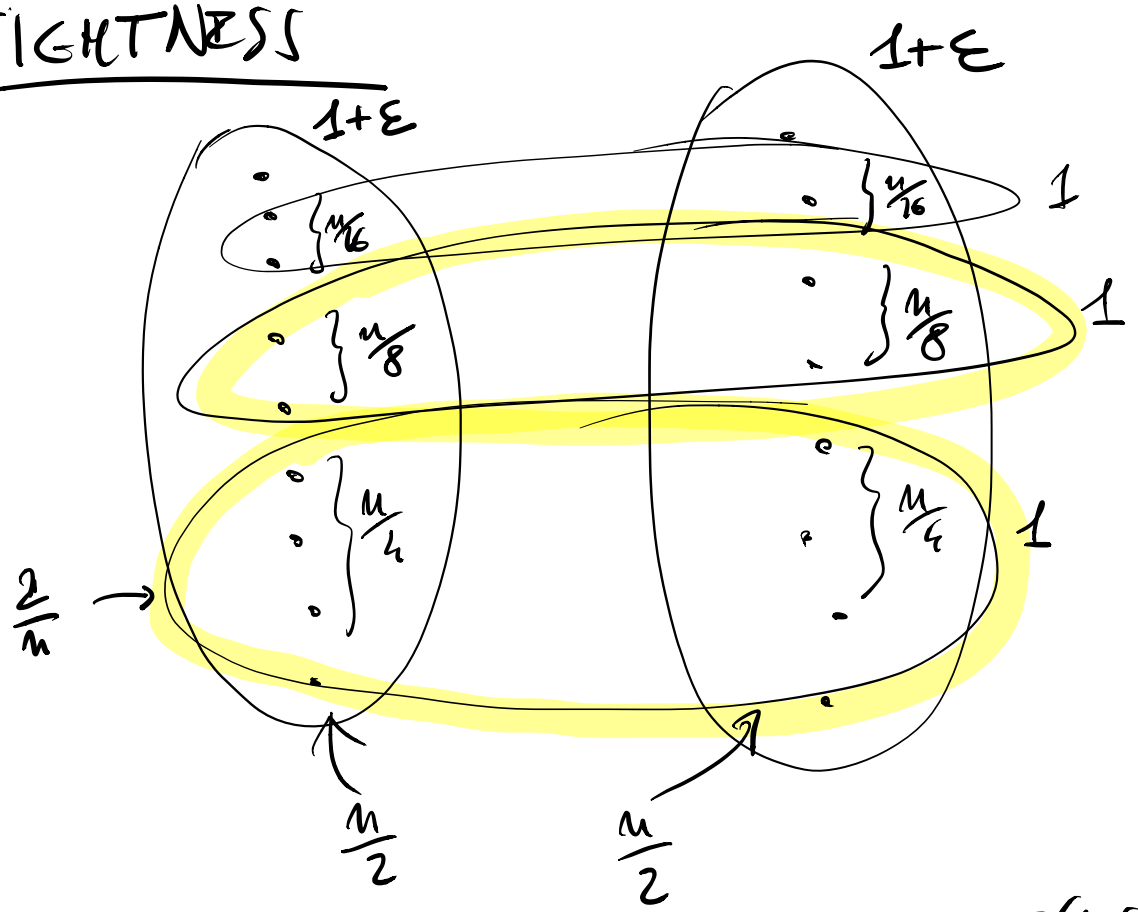
$\boxed{\text{NOTA}}$ Non si può fare meglio. Quindi se $P \neq NP$

GREEDY SET COVER $\notin$ APX

$\log(n)$-APX

$f(n)$-APX

TIGHTNESS



$$\frac{1}{n/2} = \frac{2}{n} \qquad \frac{1+\varepsilon}{n/2} = \frac{2(1+\varepsilon)}{n}$$

$$\frac{1}{n/4} = \frac{4}{n} \qquad \frac{1+\varepsilon}{\frac{n}{2}-\frac{n}{4}} = \frac{4(1+\varepsilon)}{n}$$

GREEDY $\longrightarrow$ $w = \log n$

OTTIMA $\longrightarrow$ $w^* = 2+2\varepsilon$

$$\frac{w}{w^*} = \frac{\lg n}{2+2\varepsilon}$$