Set Cover

$w_1, w_2, \ldots, w_m \in \mathbb{R}^{>0}$

$S_1, S_2, \ldots, S_m$

$\bigcup_{i=1}^{m} S_i = U \qquad |U| = n$

$\mathcal{C} \subseteq \{i_1, \ldots, i_m\}$

t.c. $\bigcup_{t=1}^{k} S_{i_t} = U$

e MIN. $w_{i_1} + \ldots + w_{i_k}$

$O(\ln n)$ - approssimazione

VERTEX COVER

INPUT: $G = (V, E)$
$w_i \in \mathbb{Q}^{>0}$ $\forall i \in V$

SOLUZIONE ACCETTABILE:
$X \subseteq V$ t.c. $\forall e \in E$ una delle due estremità di e sta in X

FUNZIONE OBIETTIVO:
$$w = \sum_{x \in X} w_x$$

TIPO: MIN

VERTEX COVER $\leq_p$ SET COVER

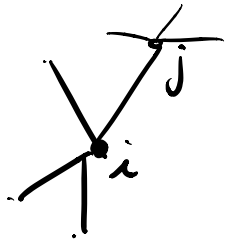
$G=(V,E)$, $w_i \in \mathbb{Q}^{>0}$ $(i \in V)$, ϑ $\xrightarrow{f}$ $S_1, \dots, S_n$, $\tilde{w}_1, \dots, \tilde{w}_n$, $\tilde{\vartheta}$

$S_i = \{e \in E \mid i \in e\}$ $U = E$

$i \in V$

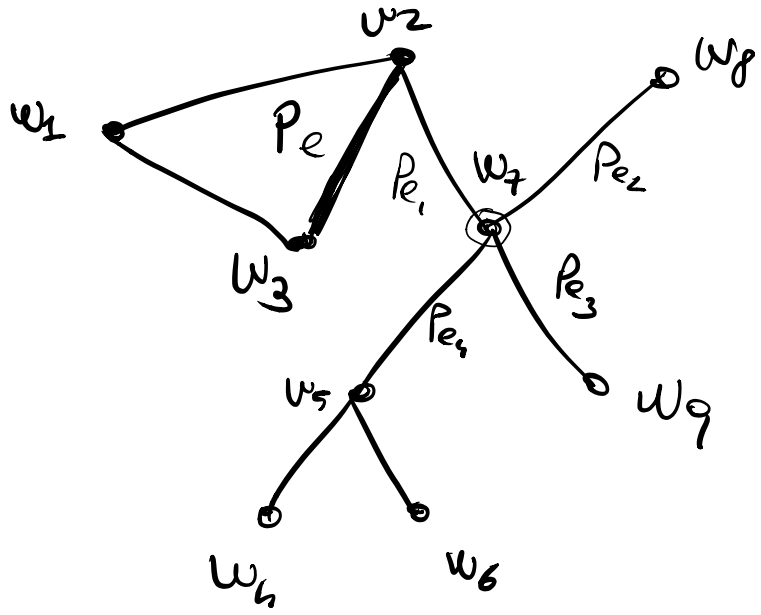
$\tilde{w}_i = w_i$

$\tilde{\vartheta} = \vartheta$



f offre una trasformazione naturale anche da VERTEX COVER a SET COVER

Corollario: VERTEX COVER è $O(\ln m)$-approssimabile



$$\forall i \quad \sum_{\substack{e \text{ t.c.} \\ i \in e}} P_e \leq w_i \qquad \text{equo}$$

Def: Sia $P_e \in \mathbb{Q}^{\geq 0}$ $(e \in E)$ un insieme di offerte. Si dice che è equo se $\forall i$

$$\sum_{\substack{e \text{ t.c.} \\ i \in e}} P_e \leq w_i$$

Lemma 1: Se p_e è ammissibile, allora

$$\sum_{e \in E} p_e \leq w^*$$

Dim: $\forall i \in V$.

$$\sum_{\substack{e \in E \\ \text{t.c. } i \in e}} p_e \leq w_i \qquad i \in S^*$$

$$\sum_{e \in E} p_e \leq \sum_{i \in S^*} \sum_{\substack{e \in E \\ \text{t.c. } i \in e}} p_e \leq \sum_{i \in S^*} w_i \leq w^*$$

$\forall e \in E$

□

p_e è **stretto** su i se

$$\sum_{\substack{e \in E \\ i \in e}} p_e = w_i.$$

PRICE VERTEX COVER

$p_e \leftarrow 0$ $\quad \forall e \in E$

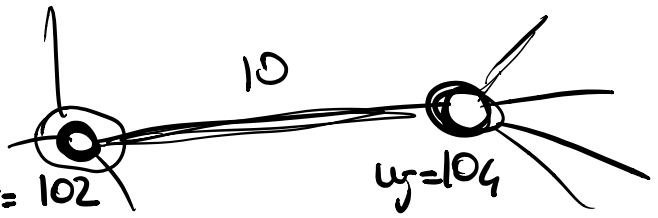
while $\exists ij \in E$ t.c. p_e non sono stretti né su i né su j

$\quad \bar{e} \leftarrow ij$ un lato di questo tipo

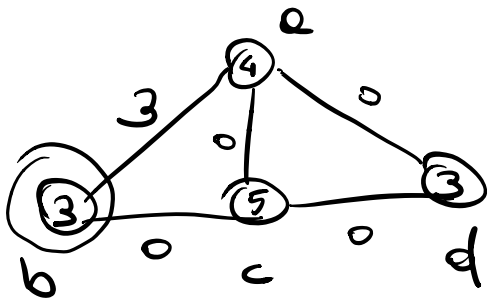
$\quad \Delta = \min\left(w_i - \sum_{\substack{e \in E \\ i \in e}} p_e \,,\; w_j - \sum_{\substack{e \in E \\ j \in e}} p_e \right)$

$\quad p_{\bar{e}} \leftarrow p_{\bar{e}} + \Delta$

Output $\{ i \mid p_e$ è stretto su $i \}$

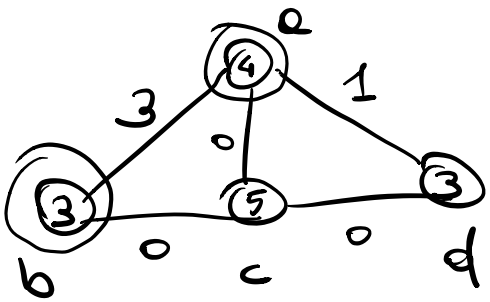


$\sum_{\substack{e \in E \\ i \in e}} p_e = 102$ $\qquad$ $\sum_{\substack{e \in E \\ j \in e}} p_e = 100$



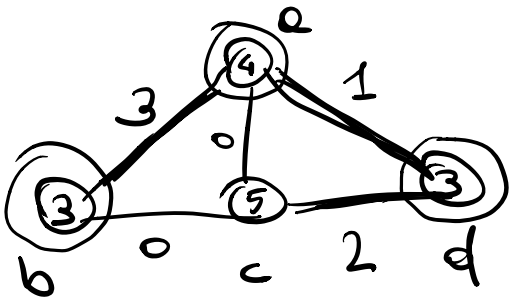
$\bar{e} = ab$

$\Delta = 3$



$\bar{e} = ad$

$\Delta = 1$



$\bar{e} = cd$

$\Delta = 2$

$\{a, b, d\}$

$4 + 3 + 3 = 10$

$3 + 3 + 1 + 1 + 2 = 10$

Lemma 2: Alla fine di PRICE SET COVER

$$w \leq 2 \sum_{e \in E} p_e$$

Dim:

$$w = \sum_{i \in S} w_i = \sum_{i \in S} \sum_{\substack{e \in E \\ i \in e}} p_e \leq$$

Stretto 22

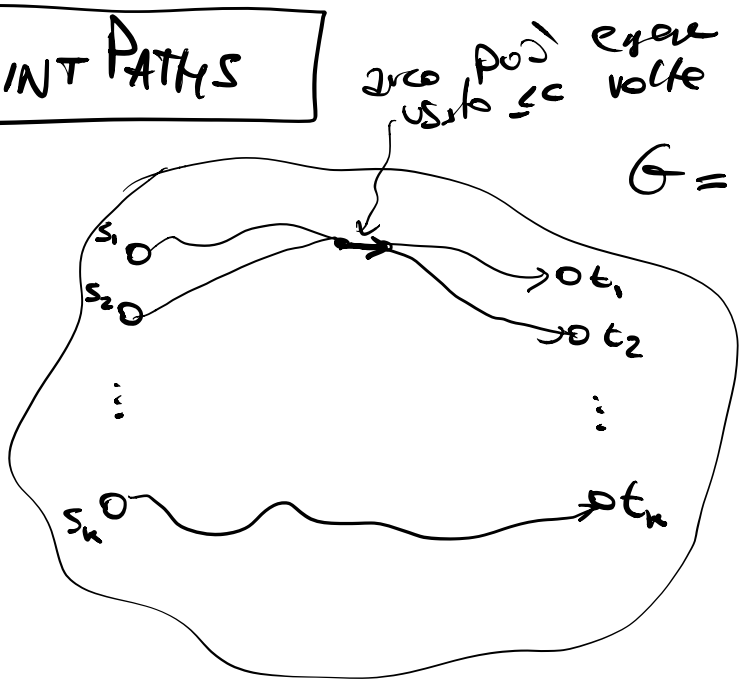
$$\leq 2 \sum_{e \in E} p_e . \qquad \square$$

Thm: PRICE VERTEX COVER è una 2-approssimazione

Dim:

$$\frac{w}{w^*} \leq \frac{2 \sum_{e \in E} p_e}{w^*} \leq \frac{2 \sum_{e \in E} p_e}{\sum_{e \in E} p_e} = 2 \qquad \square$$

DISJOINT PATHS



$G = (N, A)$
$A \subseteq N^2$

INPUT: $G = (N, A)$ orientato e $s_1, \dots, s_k, t_1, \dots, t_k \in N$
e $c \in \mathbb{N}^{>0}$

OUTPUT: $I \subseteq \{1, 2, \dots, k\}$
e $\forall i \in I$ un cammino $\pi_i : s_i \rightsquigarrow t_i$
t.c. nessun arco $a \in A$ sia usato da più di c cammini

FUNZIONE OBIETTIVO: $|I|$

TIPO: MAX

$\ell : A \to \mathbb{R}^{>0}$

$\ell(\pi = \langle x_1, x_2, \dots, x_h \rangle) =$

$= \ell(x_1, x_2) + \ell(x_2, x_3) + \dots + \ell(x_{h-1}, x_h)$

GREEDY PATHS WITH CAPACITY (parametro $\beta > 1$)

$I \leftarrow \emptyset$

$P \leftarrow \emptyset$

$\forall a \in A. \quad \ell(a) \leftarrow 1$

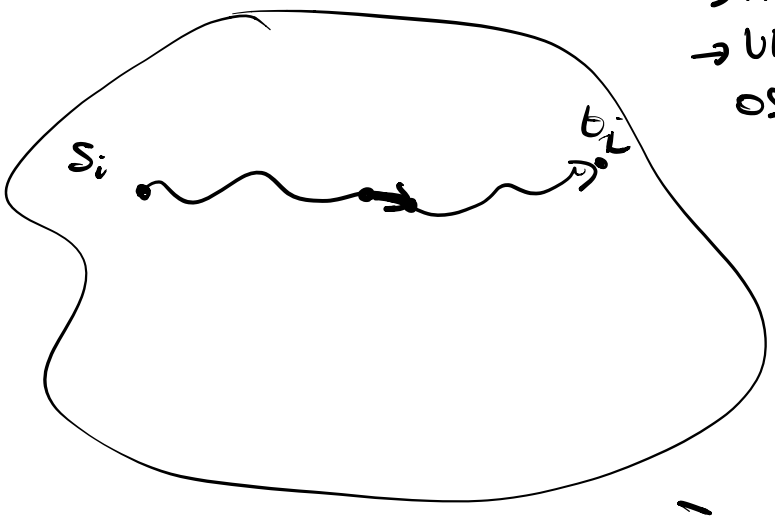
forever

- find shortest path π connecting $s_i \rightsquigarrow t_i$ (Dijkstra)
- $i \notin I$
- if $\nexists$ break
- $I \leftarrow I \cup \{i\}$
- $P \leftarrow P \cup \{\pi\}$
- for all $a \in \pi$

  $\ell(a) \leftarrow \ell(a) \cdot \beta$

  ⊛ if a is used by c paths remove a

output I, P

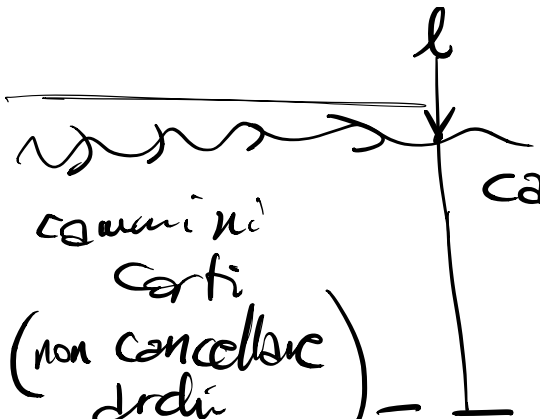


→ inutile
→ utile ma costruito

Def. Un cammino π è corto rispetto a ℓ sse $\ell(\pi) < B^c$ lungo altrimenti.

Proprietà: Finché ci sono cammini utili, non costruiti e corti, l'algoritmo sceglie uno di questi.

(a) già usato c volte $\ell(a) = B^c$



= insieme degli indici cammini in quel momento

Lemma 1: Se $i \in I^* \setminus I$. Allora

$$\bar{\ell}(\pi_i^*) \geq \beta^c$$

$$\bar{\ell}(\pi_i^*) < \beta^c$$

Dim: Se fare
- π_i^* è corto
- π_i^* è stile

} dovremo selezionarlo □

Lemma 2:

$$\sum_{\gamma \in A} \bar{\ell}(\gamma) \leq \beta^{c+1} |\bar{I}| + m.$$

Dim: All'inizio

$$\sum_{\gamma \in A} \ell(\gamma) = m$$

Immaginiamo

$$\ell_1 \xrightarrow{\pi} \ell_2$$

$\forall \gamma \in \pi$

$$\ell_2(a) = \begin{cases} \beta \ell_1(a) & \text{se } a \in \pi \\ \ell_1(a) & \text{se } a \notin \pi \end{cases}$$

$$\sum_{a \in A} \ell_2(a) - \sum_{a \in A} \ell_1(a) =$$

$$= \sum_{a \in A} \left(\ell_2(a) - \ell_1(a) \right) =$$

$$= \sum_{a \in \pi} \left(\beta \ell_1(a) - \ell_1(a) \right) =$$

$$= \sum_{a \in \pi} (\beta - 1) \ell_1(a) =$$

$$= (\beta - 1) \sum_{a \in \pi} \ell_1(a) =$$

$$= (\beta - 1) \ell_1(\pi) < (\beta - 1) \beta^c \leq \beta^{c+1}.$$

□

Osservazioni

1) $$\sum_{i \in I^* \setminus I} \bar{\ell}(\pi_i^*) \underbrace{\geq}_{\text{Lemma 1}} \beta^c \left| I^* \setminus I \right|$$

2) $$\sum_{i \in I^* \setminus I} \bar{\ell}(\pi_i^*) \leq c \sum_{a \in A} \bar{\ell}(a) \leq$$

$$\underbrace{\leq}_{\text{Lemma 2}} c \left(\beta^{c+1} |\bar{I}| + m \right)$$

Quindi

$$\beta^c |I^*| = \beta^c |I^* \setminus I| + \beta^c |I^* \cap I|$$

$$\text{Oss.1} \rightsquigarrow \leq \sum_{i \in I^* \setminus I} \ell(\pi_i^*) + \beta^c |I|$$

$$\text{Oss.2} \rightsquigarrow \leq c\left(\beta^{c+1} |\overline{I}| + m\right) + \beta^c |I|$$

$$\overline{I} \subseteq I \rightsquigarrow \leq c\left(\beta^{c+1} |I| + m\right) + \beta^c |I|.$$

Dividiamo per β^c:

$$|I^*| \leq c\beta |I| + c\beta^{-c} m + |I| \leq$$

$$\leq c\beta |I| + c\beta^{-c} m\, |I| + |I|$$

Dividendo per $|I|$:

$$\frac{|I^*|}{|I|} \leq c\beta + c\beta^{-c} m + 1 =$$

$$= c\left(\beta + \beta^{-c} m\right) + 1$$

Ponendo $\beta = m^{\frac{1}{c+1}}$

$$\frac{|I^*|}{|I|} \leq c\left(m^{\frac{1}{c+1}} + m^{\frac{-c}{c+1}} \cdot m\right) + 1 =$$

$$= c\left(m^{\frac{1}{c+1}} + m^{\frac{-c+c+1}{c+1}}\right) + 1 =$$

$= 2cm^{\frac{1}{c+1}} + 1$

c	
1	$2\sqrt{m} + 1$
2	$4\sqrt[3]{m} + 1$
3	$6\sqrt[4]{m} + 1$
...	...

Thm: GREEDY DISJOINT PATHS (con $\beta = m^{\frac{1}{c+1}}$) fornisce una $(2cm^{\frac{1}{c+1}} + 1)$- approssimazione.

k $\quad s_1, \dots, s_k$
$\quad t_1, \dots, t_k$

$k >> 2\sqrt{m}$