PROBLEMA DELLO ZAINO (KNAPSACK)

<u>INPUT</u>: $(w_0, v_0), \dots, (w_{n-1}, v_{n-1})$

$w_i, v_i \in \mathbb{N}$

$W \in \mathbb{N}$ capacità dello zaino

<u>SOL. AMMISSIBILI</u>:

$X \subseteq n$ t.c.

$$\sum_{i \in X} w_i \leq W$$

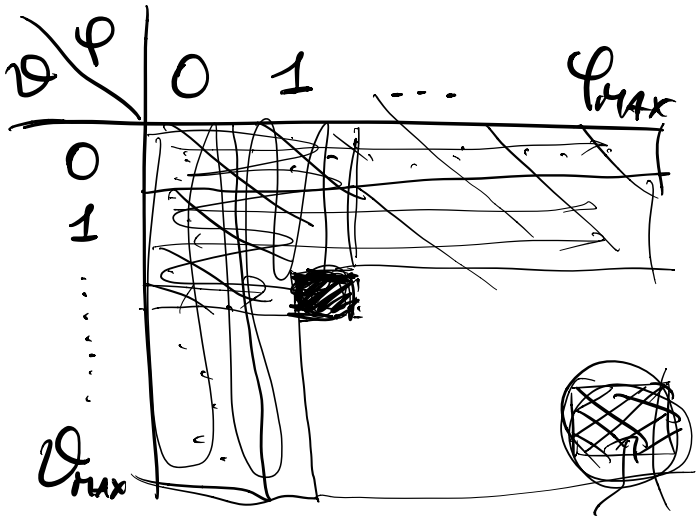
<u>FUNZ. OBIETTIVO</u>

$$\sum_{i \in X} v_i$$

<u>TIPO</u>: MAX

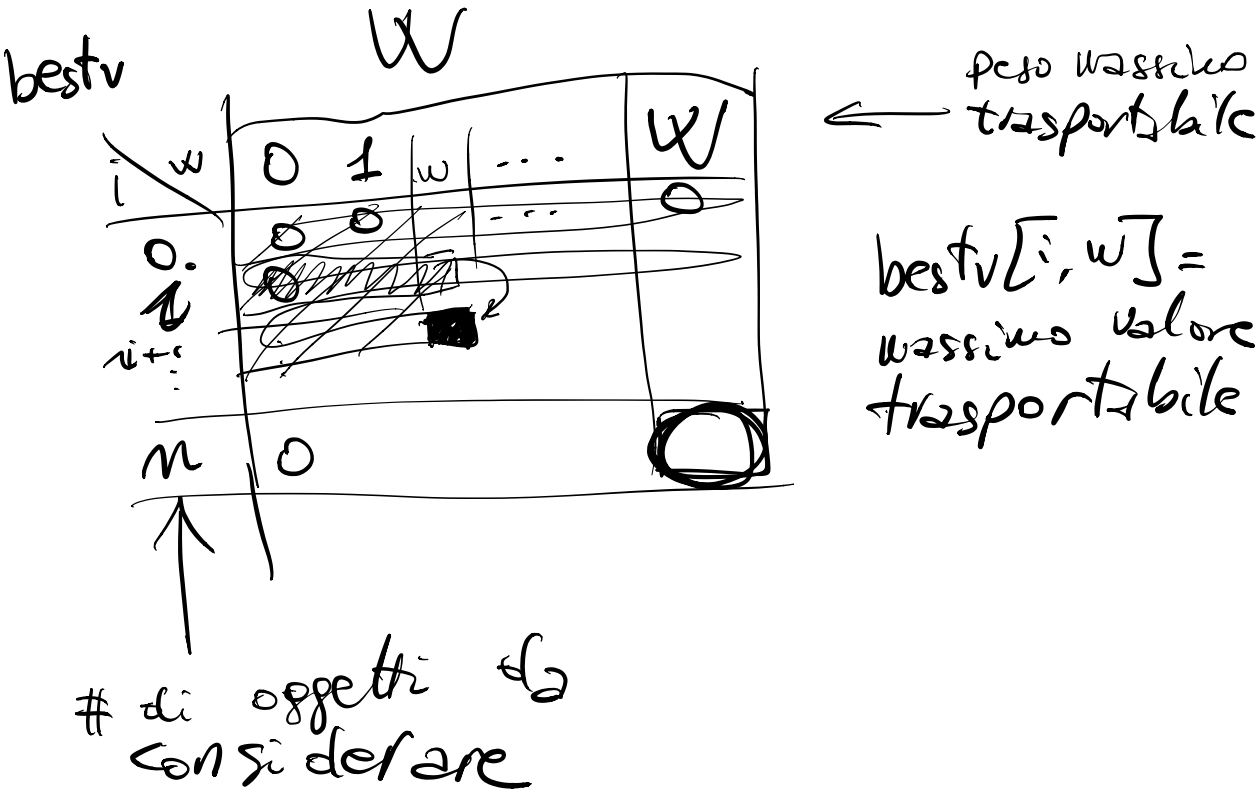
PROGRAMMAZIONE DINAMICA

Π $\qquad$ $v_1, v_2, \dots, v_k$



SOLUZIONE (A)

$$\underbrace{(w_0, v_0), \dots,}_{i} \quad (w_{n-1}, v_{n-1})$$



$$bestv[0, w] = 0 \qquad \forall w$$

$$bestv[i, 0] = 0 \qquad \forall i$$

$$bestv[i+1, w] = \begin{cases} bestv[i, w] & \text{se } w_i > w \\ \max\left(bestv[i, w],\ bestv[i, w - w_i] + v_i\right) & \text{altrimenti} \end{cases}$$

INPUT: $(w_0, v_0), \ldots, (w_{n-1}, v_{n-1})$

W

```
bestv ← MAT(n+1, W+1)
for w = 0, ..., W
    bestv[0, w] ← 0
for i = 0, ..., n
    bestv[i, 0] ← 0
for i = 1, ..., n
    for w = 1, ..., W
        if w_i > w
            bestv[i, w] ← bestv[i-1, w]
        else
            bestv[i, w] ← max(bestv[i-1, w],
                              bestv[i-1, w-w_i] + v_i)

output bestv[n, W]
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Teorema: Soluzione (A) ha complessità in tempo [e spazio] $O(nW)$. pseudopolinomiale.

$(w_0, v_0), \ldots, (w_{n-1}, v_{n-1})$

$\boxed{W}$

$|W| = \lg_2 W \Rightarrow W = 2^{|W|}$

SOLUZIONE (B)

$(w_0, v_0), \dots, (w_{n-1}, v_{n-1})$

W

bestw

$i \backslash v$	0	1	..	v	$\sum_i v_i$
0	0	$+\infty$			$+\infty$
1	0				
⋮					
n	0			$\boxed{W}$	$\boxed{\sum_i w_i}$

$\sum_i v_i \leftarrow$ voglio portare a casa $\geq v$

di oggetti da considerare

bestw$[i, v]$ = minimo peso che consente di portare a casa $\geq v$ usando solo i primi i oggetti

$$\text{bestw}[i+1, v] = \min(\text{bestw}[i, v], \text{bestw}[i, v - v_i] + w_i)$$

Thm: La soluzione (B) ha complessità in tempo [e in spazio]

$$\underbrace{O\left(M \sum_i v_i\right)} = \underbrace{O\left(\underbrace{n^2}\, \underbrace{v_{MAX}}\right)}.$$

RESCALING

$$\Pi = ((w_i, v_i), W) \qquad 0 < \varepsilon \leq 1$$

$$\vartheta = \frac{\varepsilon v_{max}}{2M}$$

$$\overline{\Pi} = ((w_i, \overline{v}_i), W) \qquad \overline{v}_i = \left\lceil \frac{v_i}{\vartheta} \right\rceil \vartheta$$

$$\hat{\Pi} = ((w_i, \hat{v}_i), W) \qquad \hat{v}_i = \left\lceil \frac{v_i}{\vartheta} \right\rceil$$

RISOLVEREMO QUESTO

Lemma 1: Le soluzioni ottime $\overline{S}^*$ e $\hat{S}^*$ di $\overline{\Pi}$ e $\hat{\Pi}$ coincidono. I valori ottimi $\overline{v}^*$ e $\hat{v}^*$ sono t.c.

$$\overline{v}^* = \hat{v}^* \vartheta$$

Lemma ②: Sia S una soluzione ammissibile di Π.
Allora

$$\sum_{i \in S} v_i \leq (1+\varepsilon) \sum_{i \in \hat{S}^*} v_i$$

Dim:

$$\sum_{i \in S} v_i \leq \sum_{i \in S} \overline{v_i} \quad \left(\text{per via del } \lceil \cdot \rceil\right)$$

$$\leq \sum_{i \in \bar{S}^*} \overline{v_i} \quad \left(\begin{array}{l} S \text{ è ammiss. per } \overline{\Pi} \\ \text{e } \bar{S}^* \text{ è ottima} \end{array}\right)$$

$$= \sum_{i \in \hat{S}^*} \overline{v_i} \quad \left(\begin{array}{l} \bar{S}^* = \hat{S}^* \\ \text{per Lemma 1} \end{array}\right)$$

$$\leq \sum_{i \in \hat{S}^*} (v_i + \vartheta) =$$

$$= \sum_{i \in \hat{S}^*} v_i + |\hat{S}^*| \vartheta \leq$$

$$\leq \sum_{i \in \hat{S}^*} v_i + \cancel{n} \frac{\varepsilon v_{MAX}}{2\cancel{n}} =$$

$$= \sum_{i \in \hat{S}^*} v_i + \frac{\varepsilon v_{MAX}}{2} \quad (*)$$

Siccome è vera per ogni soluzione ammissibile, è

vera anche $S = \{ i_{MAX} \}$.

indice che dà v_{MAX}

Applicando (*):

$$v_{MAX} \leq \sum_{i \in \hat{S}^*} v_i + \frac{\varepsilon \, v_{MAX}}{2} \leq \quad (\varepsilon \leq 1)$$

$$\leq \sum_{i \in \hat{S}^*} v_i + \frac{v_{MAX}}{2}$$

$$\Rightarrow \quad \sum_{i \in \hat{S}^*} v_i \geq \frac{v_{MAX}}{2} \quad (**).$$

Riportando (*)

$$\sum_{i \in S} v_i \leq \sum_{i \in \hat{S}^*} v_i + \frac{\varepsilon \, v_{MAX}}{2} \quad (*)$$

applicando (**)

$$\sum_{i \in S} v_i \leq \sum_{i \in \hat{S}^*} v_i + \varepsilon \sum_{i \in \hat{S}^*} v_i =$$

$$= \left(\sum_{i \in \hat{S}^*} v_i \right) (1 + \varepsilon). \quad \square$$

Osservate:

$$\hat{v}_{MAX} = \left\lceil \frac{v_{MAX}}{\vartheta} \right\rceil = \left\lceil \frac{\cancel{v_{MAX}}}{\frac{\varepsilon \cancel{v_{MAX}}}{2n}} \right\rceil = \left\lceil \frac{2n}{\varepsilon} \right\rceil \leq$$

$$\leq \frac{2n}{\varepsilon} + 1 .$$

<u>Corollario</u>: L'algoritmo Soluzione (B) $\hat{\Pi}$ applicato al problema di Knapsack fornisce una $(1+\varepsilon)$-approssim. in tempo [e spazio] $O\left(\frac{n^3}{\varepsilon}\right)$.

<u>Dim</u>: (B) trova $\hat{S}^*$

$$\boxed{(1+\varepsilon) \boxed{\sum_{i \in \hat{S}^*} v_i} \geq \boxed{\sum_{i \in S} v_i} \geq v^*}$$

lemma 2

$$1 \leq \frac{v^*}{\sum_{i \in \hat{S}^*} v_i} \leq 1+\varepsilon$$

$$\cdot\ O\left(n^2 \hat{v}_{MAX}\right) = O\left(n^2 \left(\frac{2n}{\varepsilon} + 1\right)\right) =$$

$$= O\left(\frac{n^3}{\varepsilon}\right). \qquad \square$$

<u>Corollario</u>: KNAPSACK $\in$ FPTAS.