$$\begin{cases} \text{rank}_{\underline{b}}(q) = \frac{p}{2} - \frac{1}{2} \\ \text{rank}_{\underline{b}}(q) = \frac{p}{2} - 1 \end{cases}$$

$$\Updownarrow$$

$$\text{rank}_{\underline{b}}(q) = \left\lfloor \frac{p}{2} - \frac{1}{2} \right\rfloor$$

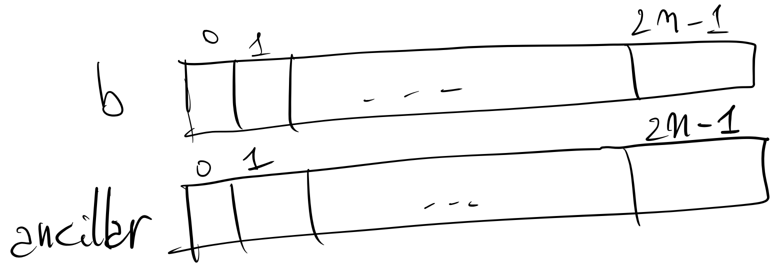
$$\text{select}_{\underline{b}}(\text{rank}_{\underline{b}}(q)) = \text{select}_{\underline{b}}\left(\left\lfloor \frac{p}{2} - \frac{1}{2} \right\rfloor\right)$$

$$\boxed{q = \text{select}_{\underline{b}}\left(\left\lfloor \frac{p}{2} - \frac{1}{2} \right\rfloor\right)}$$

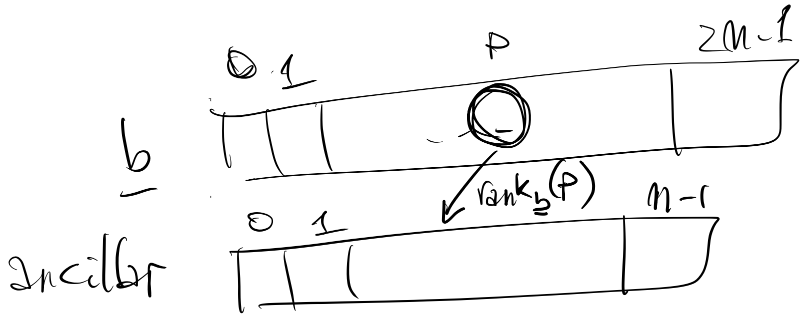
$$D_n = \underbrace{2n+1}_{\underline{b}} + \underbrace{o(2n+1)}_{\text{RANK/SELECT}} = 2n+1+o(n)$$

$$Z_n = 2n + O(\log n)$$

$$D_n - Z_n = o(n) \Rightarrow \text{SUCCINTA}$$



DATI ANC. SU TUTTI I NODI

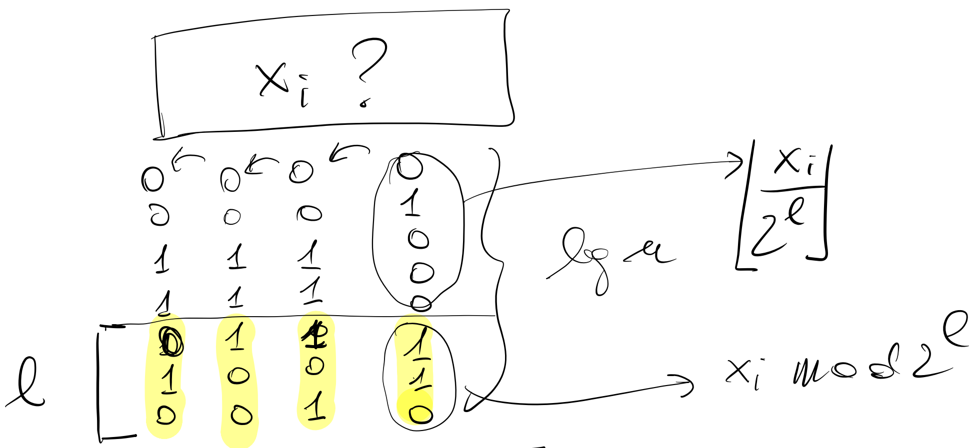


DATI ANC. SU NODI INTERNI /FOGLIE

RAPPRESENTAZIONE DI ELIAS-FANO DI SEQ. MONOTONE

$$0 \le x_0 \le x_1 \le x_2 \le \ldots \le x_{n-1} < u$$

u ↑ DIM. DELL'UNIVERSO



$$\boxed{\ell = \max\left\{0, \left\lfloor \log \frac{u}{n} \right\rfloor\right\}}$$

ASSUMO $u \ge n$ (sparsità)

PARTE INFERIORE

$$\left.\begin{aligned} \ell_0 &= x_0 \bmod 2^\ell \\ \ell_1 &= x_1 \bmod 2^\ell \\ &\vdots \\ \ell_{n-1} &= x_{n-1} \bmod 2^\ell \end{aligned}\right\}$$

ESPLICITAMENTE (USANDO ℓ BIT CIASCUNO)

PARTE SUPERIORE

$$\underbrace{u_i = \left\lfloor \frac{x_i}{2^\ell} \right\rfloor - \left\lfloor \frac{x_{i-1}}{2^\ell} \right\rfloor}_{\ldots \text{ IN UNARIO}}$$

per $i = 0, \ldots, n-1$ (si assume $x_{-1} = 0$)

MEMORIZZATI IN UNARIO
(SEQUENZE DI 0 TERMINATE DA 1)

$$u_i \quad \langle \overbrace{2,\ 0,\ 1},\ 1,\ 3 \rangle$$

SCRIVO u_i 0 SEGUITI DA UN 1

$$\begin{array}{l} \quad\ \ 0\ 1\ 2\ 3\ 4\ 5 \\ \underline{u} = \underbrace{001}_{2}\ \underbrace{1}_{0}\ \underbrace{01}_{1}\ \underbrace{01}_{1}\ \underbrace{0001}_{3} \end{array}$$

↓

RANK/SELECT

$$\text{select}_{\underline{u}}(2) = 5$$

$$\text{select}_{\underline{u}}(2) - 2 = 5 - 2 = 3$$

$\boxed{\text{SPAZIO OCCUPATO}}$

- INFERIORE (**) ℓn bit
- SUPERIORE

$$(*) = \sum_{i=0}^{n-1} (u_i + 1) = n + \sum_{i=0}^{n-1} u_i =$$

$$= n + \sum_{i=0}^{n-1} \left(\left\lfloor \frac{x_i}{2^\ell} \right\rfloor - \left\lfloor \frac{x_{i-1}}{2^\ell} \right\rfloor \right) =$$

$$= n + \left\lfloor \frac{x_{n-1}}{2^\ell} \right\rfloor - \left\lfloor \frac{x_{-1}}{2^\ell} \right\rfloor \leq n + \frac{u}{2^\ell} =$$

$$= n + \frac{u}{2^{\lfloor \log \frac{u}{n} \rfloor}}$$

Se u/n è una pot. di 2

$$(*) = n + \frac{u}{u/n} = 2n$$

Altrimenti

$$(*) \leq n + \frac{u}{2^{\log \frac{u}{n} - 1}} = n + \frac{u}{2^{\log \frac{u}{n}} \frac{1}{2}} =$$

$$= n + \frac{2u}{u/n} = 3n$$

IN TUTTO OCCUPO

$$D_n = \begin{cases} (\ell+2)\,n & \text{bit} \quad u/n \text{ pot. di } 2 \\ (\ell+3)\,n & \text{altrimenti} \end{cases}$$

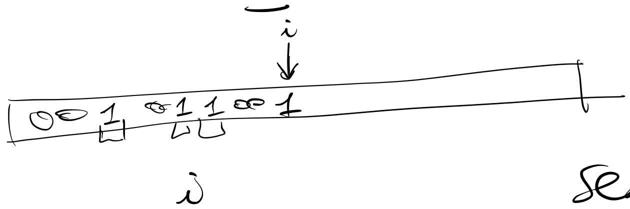
$$\ell = \left\lfloor \log \frac{u}{n} \right\rfloor$$

$$\left\lceil \log \frac{u}{n} \right\rceil = \begin{cases} \ell & \text{se } u/n \text{ è una pot. di } 2 \\ \ell+1 & \text{altr.} \end{cases}$$

$$\boxed{D_n = \left(2 + \left\lceil \log \frac{u}{n} \right\rceil\right) n + \overbrace{o(n)}^{\text{SELECT}}}$$

$$\frac{x_n}{2^\ell} \le \frac{u}{2^{\log\frac{u}{n}}} = \frac{u}{u/n} = n$$

$\text{select}_{\underline{u}}(i)$ = posizione dell'i-esimo 1 in $\underline{u}$



$$\text{select}_{\underline{u}}(i) = \underbrace{i}_{\text{UNI}} + \underbrace{u_0 + u_1 + \dots + u_i}_{\text{ZERI}}$$

$$\text{select}_u(i) - i = u_0 + u_1 + \dots + u_i =$$

$$= \sum_{j=0}^{i} \left\lfloor \frac{x_j}{2^\ell} \right\rfloor - \left\lfloor \frac{x_{j-1}}{2^\ell} \right\rfloor =$$

$$= \left\lfloor \frac{x_i}{2^\ell} \right\rfloor$$

$$x_i = \left\lfloor \frac{x_i}{2^\ell} \right\rfloor 2^\ell + x_i \bmod 2^\ell =$$

$$= (\text{select}_u(i) - i) 2^\ell + \ell_i$$

$O(1)$ $O(1)$

INF. THEORETICAL LOWER BOUND

(A) $\boxed{0 \le x_0 \le x_1 \le x_2 \le \dots \le x_{n-1} < u}$

QUANTE SONO? (per n e u fissati)

(B) Sono in biiezione con i multiinsiemi di card. n sottoinsiemi di $\{0, 1, \dots, u-1\}$

(C) $c_0, c_1, \dots, c_{u-1}$ n° di occorrenze del valore $0, 1, \dots, u-1$ nel multiinsieme

numero di soluzioni intere non-negative

(D) dell'equazione

$$c_0 + c_1 + \dots + c_{u-1} = M$$

$\boxed{\text{ESEMPIO}}$

$u = 7$ $\{0, 1, 2, \dots, 7\}$

$M = 5$

$$\begin{array}{ccccccc} c_0 + & c_1 + & c_2 + & c_3 + & c_4 + & c_5 + & c_6 = 5 \\ | & | & | & | & | & | & | \\ 0 & 1 & 0 & 2 & 0 & 1 & 1 \end{array}$$

$\{1, 3, 3, 5, 6\}$

$0 \le \boxed{1 \le 3 \le 3 \le 5 \le 6} \not\le 7$

| * || * * || * | *

STARS & BANGS

$$c_0 + c_1 + \dots + c_{u-1} = M$$

$\begin{cases} N^o \text{ SOL.} \\ \text{EQ.} \end{cases}$

$u-1$ barrette

* * || * * | * | * ||| * *

$\begin{cases} N^o \\ \text{STRINGHE} \end{cases}$

m stelline

$$\binom{u+m-1}{u-1} = \frac{(u+m-1)!}{(u-1)!\,m!} = \binom{u+m-1}{m}$$

INF. TH. LOWER BOUND

se $m \ll u$

$$Z_m = \log\binom{u+m-1}{m}$$

$$\boxed{\log\binom{A}{B} \approx B\log\frac{A}{B} + (A-B)\log\frac{A}{A-B}}$$

$$Z_m \approx m\log\frac{u+m-1}{m} + \boxed{(u-1)\log\frac{u+m-1}{u-1}} =$$

$$= m\log\frac{u+m-1}{m} =$$

$$= m\log\left(\frac{u}{m}\left(1+\frac{m}{u}-\frac{1}{u}\right)\right) =$$

$$= m\log\frac{u}{m} + m\log\left(1+\frac{m}{u}-\frac{1}{u}\right)$$

$$\boxed{x \approx \ln(1+x)}$$

$$Z_m = m\log\frac{u}{m} + m\ln\left(1+\frac{m}{u}-\frac{1}{u}\right)\frac{1}{\ln 2}$$

$$= m\log\frac{u}{m} + m\left(\frac{m}{u}-\frac{1}{u}\right)\frac{1}{\ln 2} =$$

$$= n \log \frac{u}{n} + \frac{1}{u}\frac{1}{\log 2} - \frac{n}{u}\frac{1}{\ln 2}$$

$$Z_n \approx n \log \frac{u}{n}$$

$$D_n = 2n + n \left\lceil \log \frac{u}{n} \right\rceil + o(n)$$

---

n° bit per elemento

$$\overline{Z}_n \approx \log \frac{u}{n}$$

$$\overline{D}_n = \underbrace{2} + \left\lceil \log \frac{u}{n} \right\rceil + \underbrace{o(n)} =$$

$$= \overline{Z}_n + O(1)$$

QUASI IMPLICITA (NEL CASO SPARSO)

$$n \leq \sqrt{u}$$