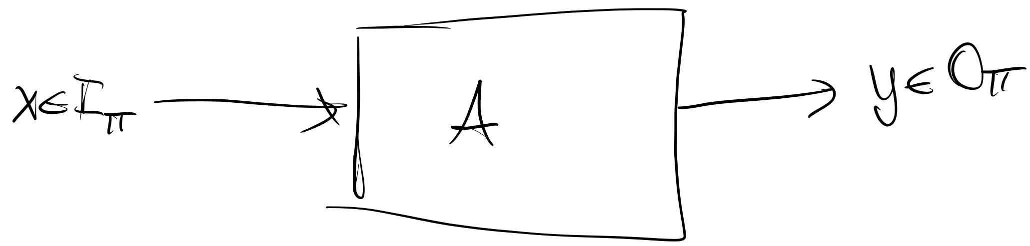
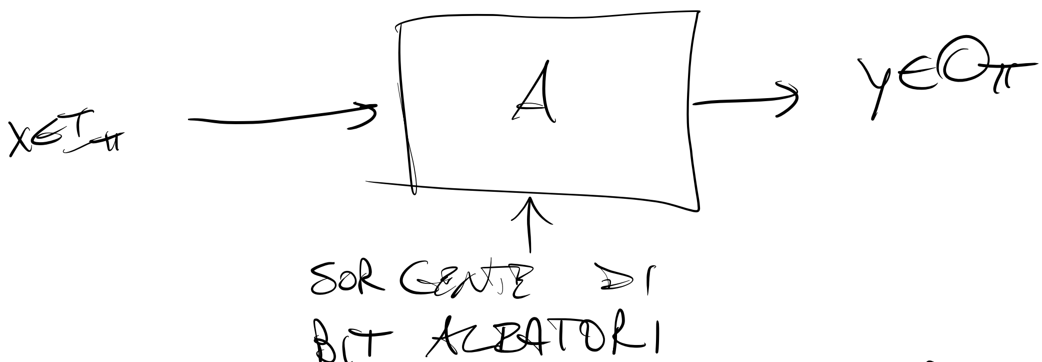
MODELLO PROBABILISTICO



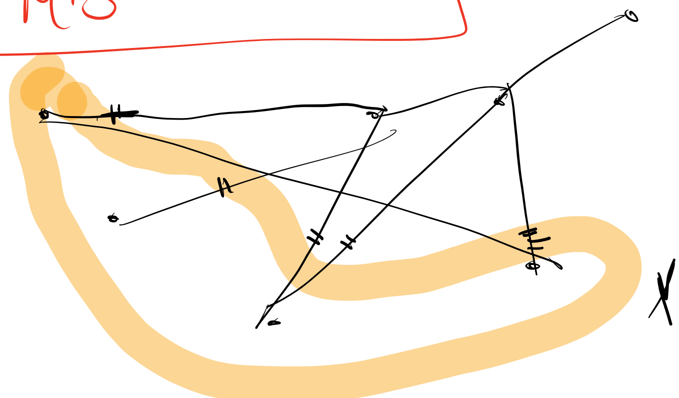


$P(T = t \mid |x| = n)$

$P(y \mid x)$

ALGORITMI
- $\boxed{\text{Montecarlo}}$
- Las Vegas

PROBLEMA DEL TAGLIO MINIMO



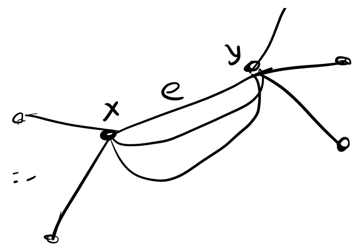
Input: $G = (V, E)$ non orientato

Sol. Amm.: $X \subseteq V$ non vuoto e $X \neq V$

Funz. obiettivo: $\left|\{e \in E \mid e \cap X \neq \emptyset \text{ e } e \setminus X \neq \emptyset\}\right|$

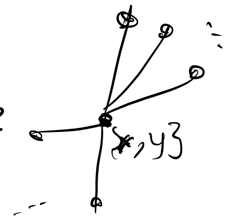
Tipo: MIN

ALGORITMO DI KARGER



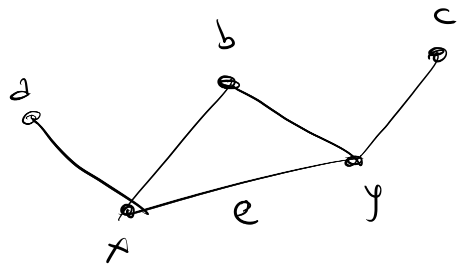
G

$G \downarrow e$

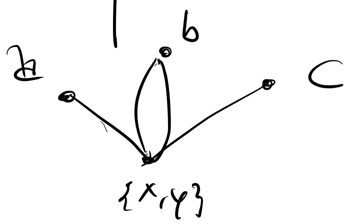


contrazione di e

G



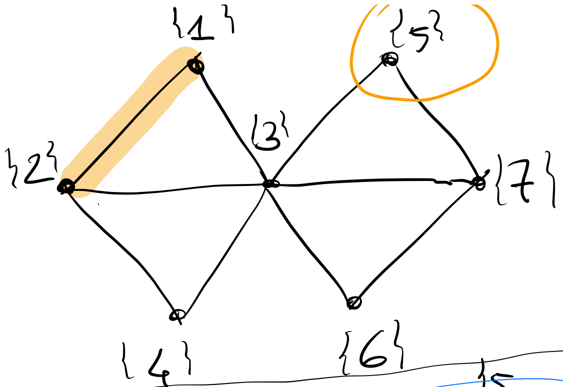
$\longrightarrow G \downarrow e$



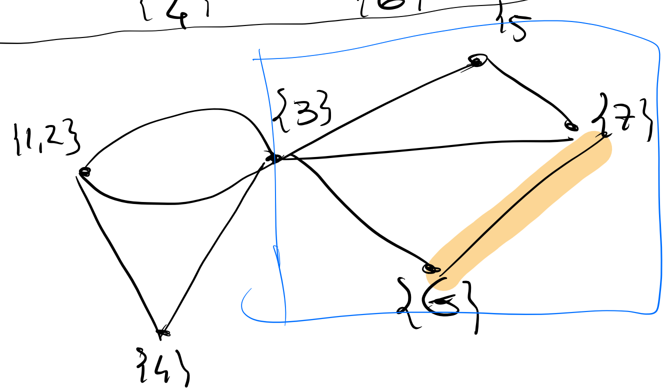
Input: $G = (V, E)$

if G non è connesso
  output una qualunque componente

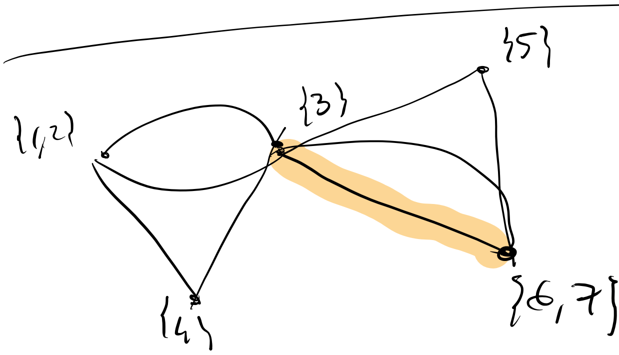
while $|V| > 2$ { uniformemente a caso
  $e \leftarrow E$
  $G \downarrow e$
}
output uno dei due vertici



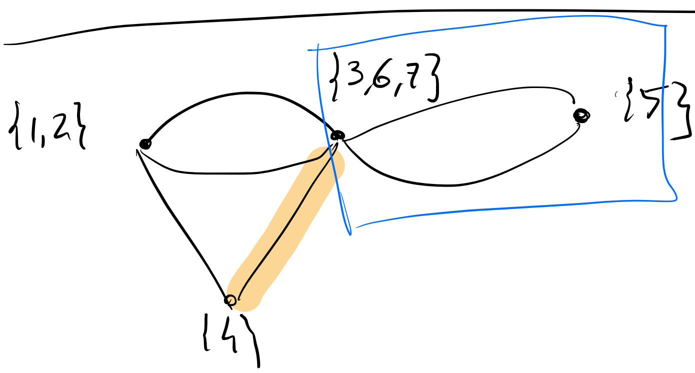
G_0



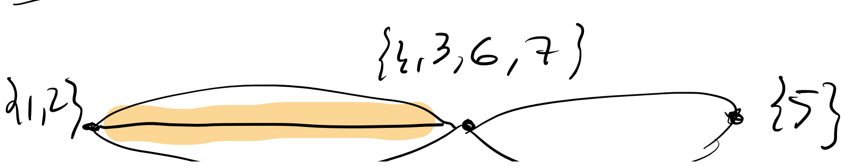
G_1



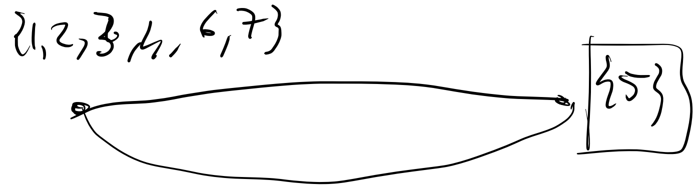
G_2



G_3



G_4



G_5

G_i = grafo dopo i passi di esecuzione

Oss. 1

- #vertici di $G_i = n-i$
- #lati di $G_i \le n-i$

Oss. 2

Ogni taglio di G_i è un taglio di G con lo stesso costo

Oss. 3

Il grado di ogni vertice di $G_i \geq k^*$

$$2(\#\text{lati di } G_i) = \sum_{v \in V_i} d_i(v) \underset{\text{Oss. 3}}{\geq} \underset{\text{Oss. 1}}{|V_i|} \cdot k^* = (n-i)\,k^*$$

$$\#\text{lati di } G_i \geq \frac{(n-i)\,k^*}{2}$$

$\mathcal{E}_i$ = all'iterazione i-esima ($G_i \to G_{i+1}$) non viene contratto un lato del taglio minimo

<u>Lemma</u>: $P[\mathcal{E}_i \mid \mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{i-1}] \geq \frac{n-i-2}{n-i} \quad \forall i \geq 0$

<u>Dim</u>: $P[\mathcal{E}_i \mid \mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{i-1}] =$

$$= 1 - P[\neg \mathcal{E}_i \mid \underbrace{\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{i-1}}_{G_0 \rightsquigarrow G_i}] =$$

$$= 1 - \frac{k^*}{\#\text{lati di } G_i} \geq$$

$$\geq 1 - \frac{2k^*}{(n-i)\,k^*} = 1 - \frac{2}{n-i} =$$

$$= \frac{n-i-2}{n-i}. \qquad \square$$

$$P[\mathcal{E}_0 \wedge \mathcal{E}_1 \wedge \mathcal{E}_2 \wedge \dots \wedge \mathcal{E}_{n-3}] =$$

$$= P[\mathcal{E}_0]\, P[\mathcal{E}_1 \mid \mathcal{E}_0]\, P[\mathcal{E}_2 \mid \mathcal{E}_0, \mathcal{E}_1] \dots P[\mathcal{E}_{n-3} \mid \mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{n-4}] \geq$$

$$\geq \boxed{\frac{n-2}{n}} \cdot \frac{n-3}{n-1} \, \frac{n-4}{n-2} \cdots \frac{n-(n-3)-2}{n-(n-3)} =$$

$$= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \, \frac{n-4}{n-2} \cdots \frac{1}{3} =$$

$$= \frac{(n-2)!}{n!/2} = \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}}$$

$$P[\mathcal{E}_0] \geq \frac{n-2}{n}$$ DA DIM.

$$P[\mathcal{E}_0] = 1 - P[\neg\mathcal{E}_0] = 1 - \frac{k^*}{\#\text{lati di } G} \geq 1 - \frac{\cancel{k^*}2}{n\cancel{k^*}} =$$

$$= 1 - \frac{2}{n} = \frac{n-2}{n}$$

Thm: L'algoritmo di ~~Karper~~ non tocca il taglio minimo con probab. $\geq \frac{1}{\binom{n}{2}}$

Corollario: Eseguendo ~~Karper~~ $\binom{n}{2} \ln n$ volte e prendendo la soluzione ottima, otteniamo il taglio minimo con probabilità $\geq 1 - \frac{1}{n}$

Dim: In una esecuzione <u>non</u> trova l'ottimo con prob. $\leq 1 - \frac{1}{\binom{n}{2}}$.

$$\leq \left(1 - \frac{1}{\binom{n}{2}}\right)^{\binom{n}{2} \ln n} \underset{\text{DEF.}}{\leq} \left(\frac{1}{e}\right)^{\ln n} = \frac{1}{n}.$$

$\square$