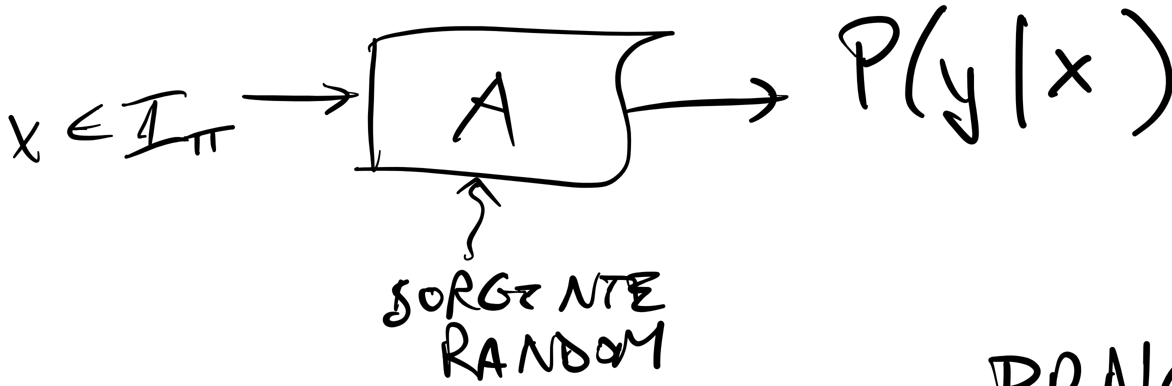
$x \in I_\pi \longrightarrow \boxed{A} \longrightarrow y \in O_\pi$



PRNG

PROBLEMA DEL TAGLIO MINIMO GLOBALE (MINCUT)

__INPUT:__ $G=(V,E)$ grafo non orientato

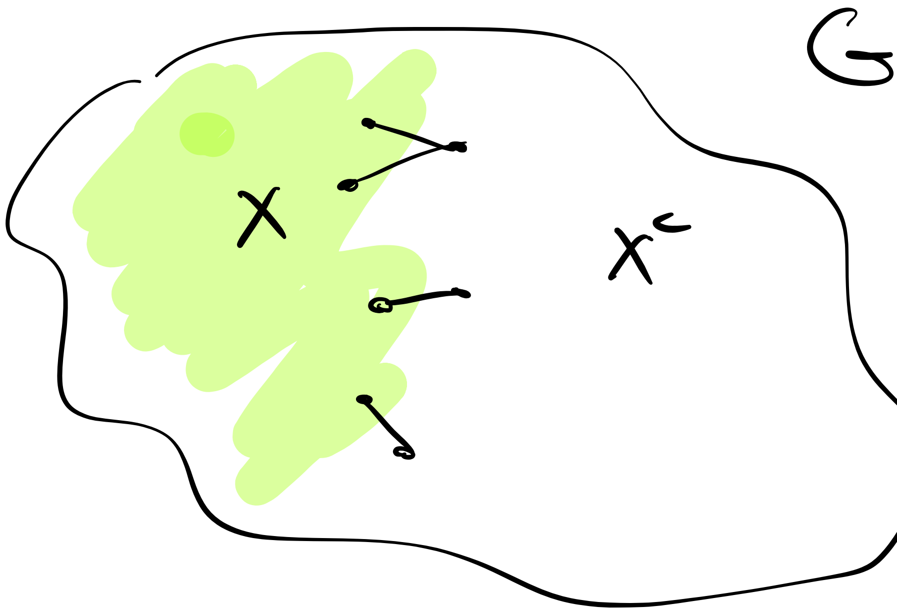
__SOL. AMMISSIBILE:__ $X \subseteq V$

$X \neq \emptyset$ $\quad X^c \neq \emptyset$

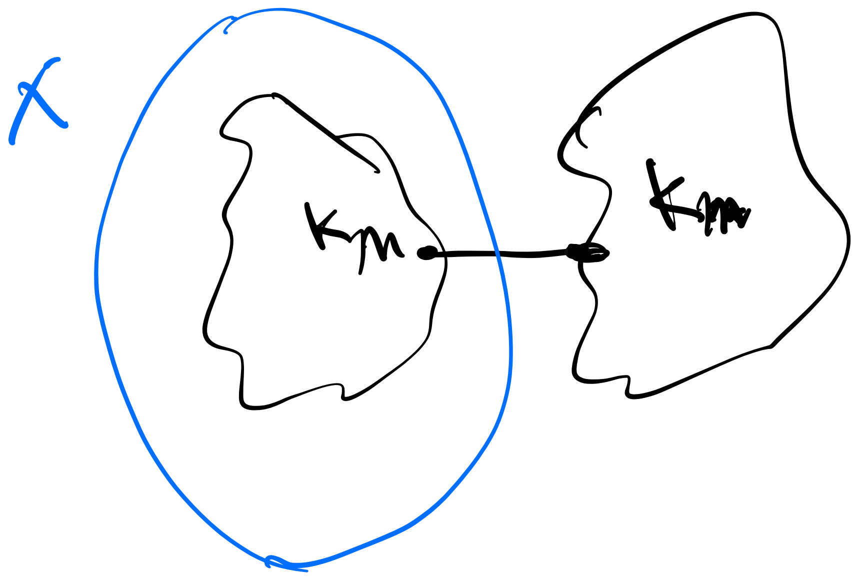
__FUNZ. OBIETTIVO:__

$\{e \in E \mid e \cap X \neq \emptyset,\ e \cap X^c \neq \emptyset\}$

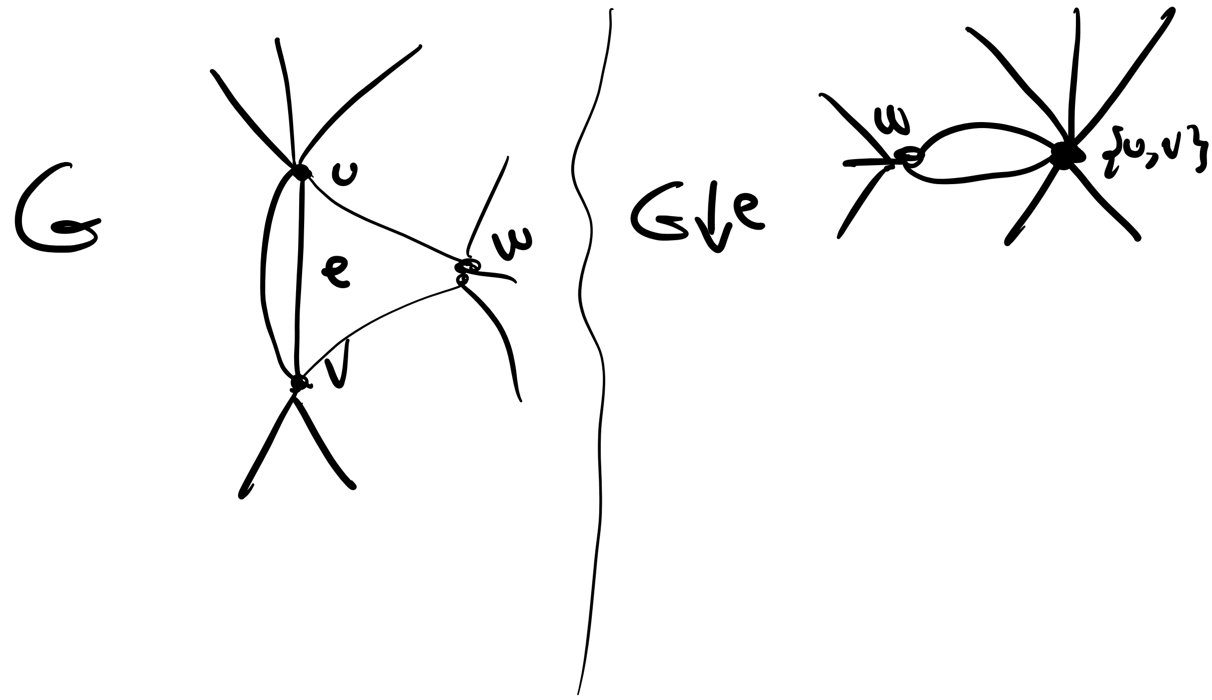
__TIPO:__ MIN



Lemma (*): Taglio minimo ≤ grado minimo



ALGORITMO DI KARGER

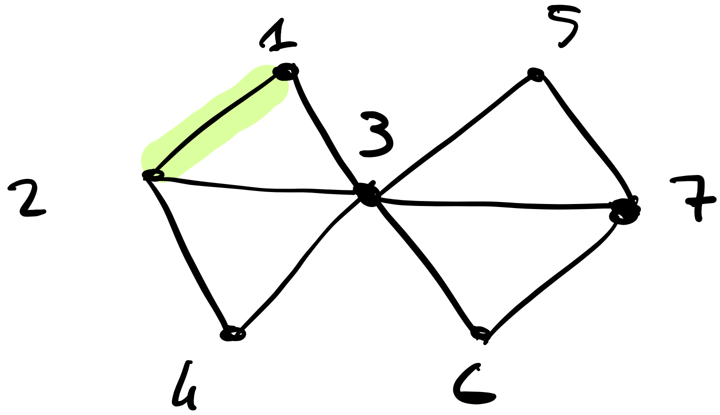


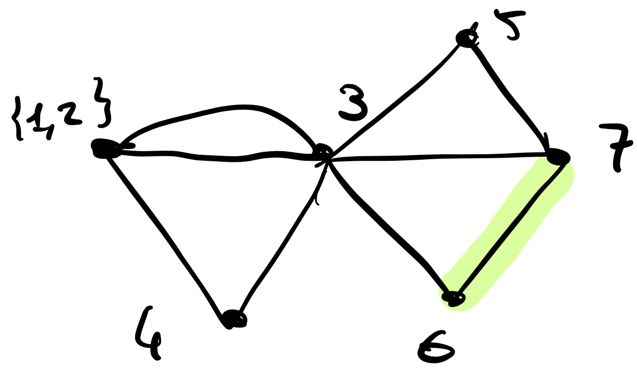
INPUT: $G=(V,E)$

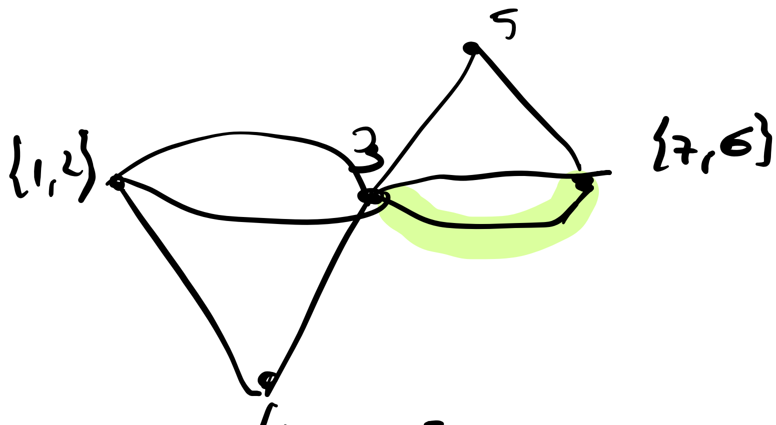
- Se G non è connesso, emetti una componente connessa qualunque
- Altrimenti

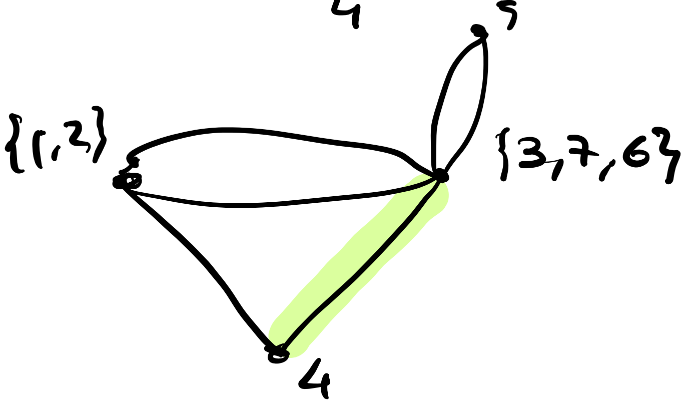
  while $|V|>2$
  - scegli un lato e a caso
  - $G\downarrow e$
- Emetti la classe di equivalenza di uno dei due vertici

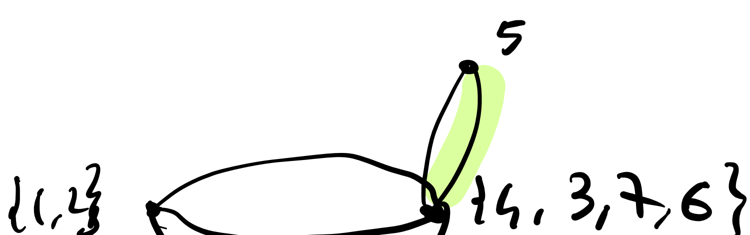
ESEMPIO

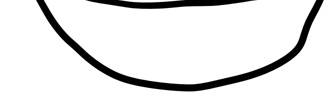


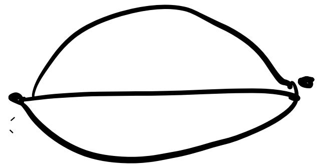






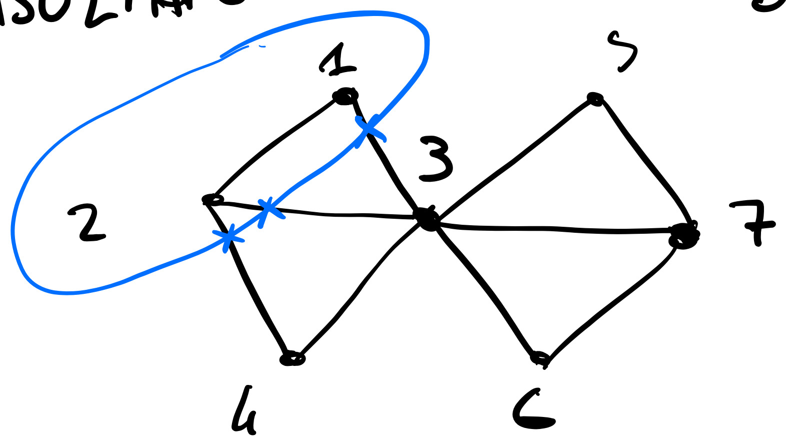




$\{1,2\}$  $\{4,5,3,7,6\}$

RISULTATO

DIMENSIONE = 3



ANALISI

$$G = G_1 \xrightarrow{G_1 \downarrow e_1} G_2 \xrightarrow{G_2 \downarrow e_2} G_3 \xrightarrow{G_3 \downarrow e_3} \dots$$

X^* taglio minimo
K^* = DIM. taglio minimo

G_i il grafo prima dell'i-esima iterazione

1) G_i ha $n-i+1$ vertici
   G_i ha $\leq m-i+1$ lati
2) Ogni taglio di G_i corrisponde a un taglio di G con la stessa dimensione
3) Grado minimo di G_i $\geq k^*$

$$2\overset{\text{lati di } G_i}{m_i} = \underbrace{\sum_{v \in V_{G_i}}}_{n-i+1 \ (1)} d_{G_i}(v) \geq k^*(n-i+1)$$

$$\boxed{m_i \geq \frac{k^*(n-i+1)}{2}}$$

$\mathcal{E}_i$ = "all'i-esima iterazione <u>non</u> contraiamo uno dei lati tagliati dal taglio minimo"

<u>Lemma</u>: $P[\mathcal{E}_i \mid \mathcal{E}_1, \dots, \mathcal{E}_{i-1}] \geq \frac{n-i-1}{n-i+1}$

<u>Dim</u>: $P[\mathcal{E}_i \mid \mathcal{E}_1, \dots, \mathcal{E}_{i-1}] =$

$$= 1 - P[\overline{\mathcal{E}_i} \mid \mathcal{E}_1, \dots, \mathcal{E}_{i-1}] =$$

$$= 1 - \frac{k^*}{m_i} \geq$$

$$\geq 1 - \frac{\cancel{k^*}\, 2}{\cancel{k^*}(n-i+1)} = \frac{n-i+1-2}{n-i+1} =$$

$$= \frac{n-i-1}{n-i+1}.$$

$\square$

Teorema: L'algoritmo di Karger emette il taglio minimo con probabilità $\geq \frac{1}{\binom{n}{2}}$.

Dim: $P[\mathcal{E}_1 \wedge \mathcal{E}_2 \wedge \mathcal{E}_3 \wedge \dots \wedge \mathcal{E}_{n-2}] =$

$$= P[\mathcal{E}_1] \cdot P[\mathcal{E}_2 \mid \mathcal{E}_1] \cdot P[\mathcal{E}_3 \mid \mathcal{E}_1, \mathcal{E}_2] \cdot \dots \cdot P[\mathcal{E}_{n-2} \mid \mathcal{E}_1, \dots, \mathcal{E}_{n-3}] \geq$$

$$\geq \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdot \dots \cdot \frac{1}{3} =$$

$$= \frac{(n-2)!\, 2}{n!} = \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}}$$

$\square$

Corollario: Eseguendo l'algoritmo di Karger $\binom{n}{2} \ln n$ volte si ottiene il taglio minimo con probabilità $\geq 1 - \frac{1}{n}$ ("quasi certamente")

Dim: $\forall x > 1$

$$\frac{1}{4} \leq \underbrace{\left(1 - \frac{1}{x}\right)^x} \leq \frac{1}{e}$$

Qual è la probabilità di **non** trovare l'ottimo?

$$\leq \left(1 - \frac{1}{\binom{n}{2}}\right)^{\binom{n}{2}\ln n} \leq \left(\frac{1}{e}\right)^{\ln n} = \frac{1}{n}$$

$\square$