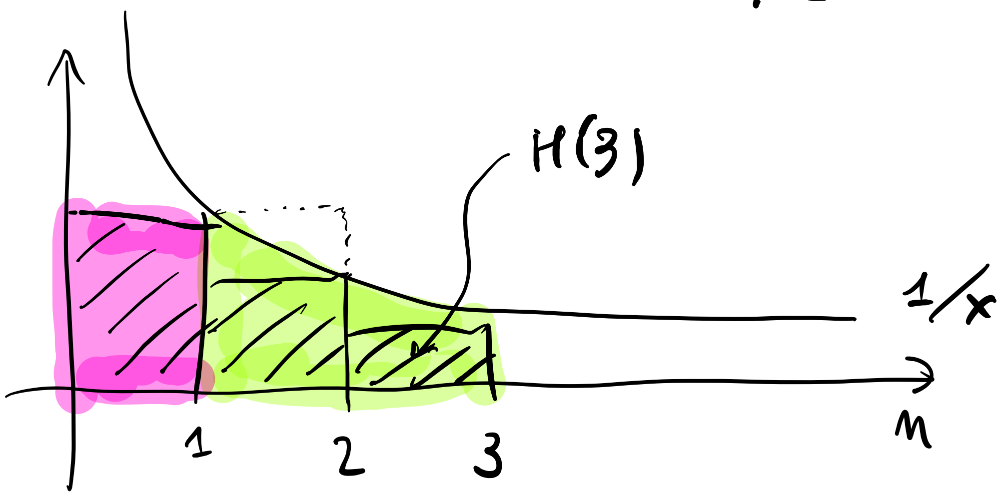
PROBLEMA DI SET COVER

FUNZIONE ARMONICA

$$H : \mathbb{N}^+ \to \mathbb{R}$$

$$H(n) = \sum_{i=1}^{n} \frac{1}{i}$$



Proprietà 1 : $H(n) \leq 1 + \ln n$

Dim :

$$H(n) \leq 1 + \int_1^n \frac{1}{x}\,dx =$$

$$= 1 + \left[\ln x\right]_1^n =$$

$$= 1 + \ln n - \ln 1 =$$

$$= 1 + \ln n \qquad \square$$

Proprietà 2: $\ln(n+1) \leq H(n)$

Dim.:

$$\int_t^{t+1} \frac{1}{x}\,dx \leq \int_t^{t+1} \frac{1}{t}\,dt = \frac{1}{t} \quad (*)$$

$$H(n) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \geq$$

$$\geq \int_1^2 \frac{1}{x}\,dx + \int_2^3 \frac{1}{x}\,dx + \dots + \int_n^{n+1} \frac{1}{x}\,dx =$$

$$= \int_1^{n+1} \frac{1}{x}\,dx = \Big[\ln x\Big]_1^{n+1} = \ln(n+1) \qquad \square$$

$$\boxed{\ln(n+1) \leq H(n) \leq 1 + \ln n}$$

SET COVER

INPUT: $S_1, S_2, \dots, S_m \subseteq \Omega$ t.c. $\bigcup_{i=1}^{m} S_i = \Omega$ $\quad n = |\Omega|$

$w_1, w_2, \dots, w_m \in \mathbb{Q}^+$

SOL. AMM.: $I \subseteq \{1, \dots, m\}$ t.c. $\bigcup_{i \in I} S_i = \Omega$

FUNZ. OBIETTIVO: $W = \sum w_i$

$i \in I$

TIPO: MIN

Teorema: $\text{SET COVER} \in \text{NPO}_C$

ALGORITMO PRICING SET COVER

$R \leftarrow \Omega$ # Punti da coprire

$I \leftarrow \emptyset$

while $R \neq \emptyset$

- choose i minimizing $\dfrac{w_i}{|S_i \cap R|} \to c_s \quad \forall s \in S_i \cap R$
- $I \leftarrow I \cup \{i\}$
- $R \leftarrow R \setminus S_i$

output I

Teorema 1: $$w = \sum_{s \in \Omega} c_s$$

Dim.:

$$w = \sum_{i \in I} w_i = \underbrace{\sum_{i \in I} \sum_{s \in S_i \cap R} c_s}_{} = \sum_{s \in \Omega} c_s$$

Teorema (2): Per ogni K

$$\sum_{s \in S_K} c_s \leq H(|S_K|) \cdot w_K$$

Dim:

$$S_K = \{\underbrace{s_1, s_2, \dots, s_d}_{\text{in ordine di copertura}}\}$$

Per ogni j, ci sarà un'iterazione in cui viene scelto un $\bar{K}$ che copre s_j.

$$R \supseteq \{s_j, \dots, s_d\}$$

$$|S_K \cap R| \geq d - j + 1$$

$$c_{s_j} = \frac{w_{\bar{K}}}{|S_{\bar{K}} \cap R|} \overset{\text{min.}}{\leq} \frac{w_K}{|S_K \cap R|} \leq \frac{w_K}{d-j+1}.$$

Quindi

$$\sum_{s \in S_K} c_s \leq \sum_{j=1}^{d} \frac{w_K}{d-j+1} =$$

$$= w_K \sum_{j=1}^{d} \frac{1}{d-j+1} =$$

$$= w_K \left(\frac{1}{d} + \frac{1}{d-1} + \frac{1}{d-2} + \dots + \frac{1}{1} \right) =$$

$$= W_k \quad H(d) = W_k H(|S_k|) \qquad \square$$

Chiamo

$$M = \max_k |S_k|$$

<u>Teorema:</u> L'algoritmo PRICING SET COVER è $H(M)$- approssimazione.

<u>Dim</u>: Sia

$$w^* = \sum_{i \in I^*} w_i .$$

Per il Teorema 2

(A) $\forall k \quad W_k \geq \dfrac{\sum_{s \in S_k} c_s}{H(|S_k|)} \geq \dfrac{\sum_{s \in S_k} c_s}{H(M)}$.

(B) $$\underbrace{\sum_{i \in I^*} \sum_{s \in S_i}}_{I^* \text{ è ammissibile}} c_s \geq \sum_{s \in \Omega} c_s \underset{\substack{\uparrow \\ \text{Teorema 1}}}{=} W$$

$$w^* = \sum_{i \in I^*} w_i \underset{(A)}{\geq} \frac{\sum_{i \in I^*} \sum_{s \in S_i} c_s}{H(M)} \underset{(B)}{\geq} \frac{W}{H(M)}$$

$$\Rightarrow \frac{W}{W^*} \le H(M).$$

$\square$

Corollario: Asintoticamente, PRICING SET COVER è una $O(\ln \underbrace{n}_{|\Omega|})$-approssimazione

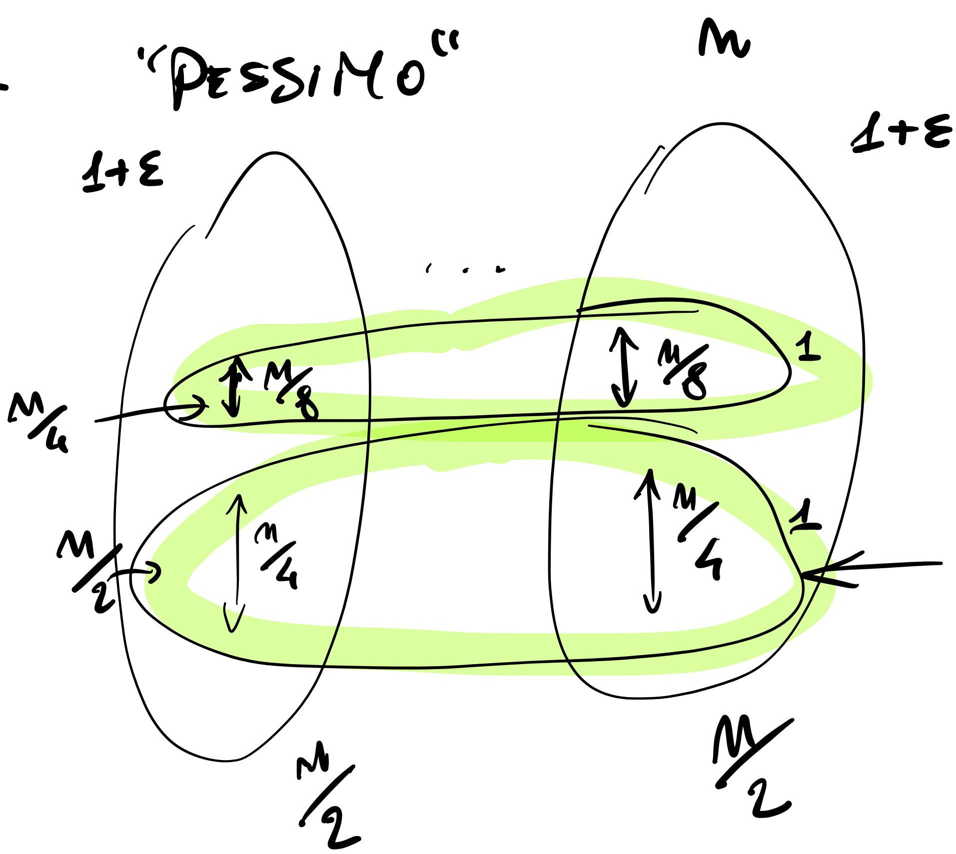
Dim.:

$$M = \max_k |S_k| \le |\Omega| = n$$

$$H(M) \le H(n) = O(\ln n) \qquad \square$$

STRETTEZZA DELL'ANALISI

INPUT "PESSIMO"



$$\boxed{1} \rightarrow \frac{1+\varepsilon}{n/2} \qquad \frac{1}{n/2}$$

$$1 \rightarrow \frac{1+\varepsilon}{\frac{n}{2}-\frac{n}{4}} = \frac{1+\varepsilon}{n/4} \qquad \frac{1}{n/4}$$

$$\boxed{W = 1 \cdot \log n}$$

$$\boxed{W^* = 2+2\varepsilon}$$

$$\frac{W}{W^*} = \Omega(\log n)$$

TEOREMA

- SET COVER $\in \log$-APX
- Se $P \neq NP$
  SET COVER $\notin$ APX