Triangle counting

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Triangles and clustering

One of the distinguishing features of many complex (undirected) networks is their relatively large *clustering coefficient*:

- the clustering coefficient of a vertex x is the fraction of its pairs of neighbors that are neighbors of each other
- i.e., the fraction of triples (y_1, x, y_2) formed by two edges that form themselves a triangle.
- Social networks exhibit a relatively large clustering coefficient, compared to their diameter.

Local vs. global clustering coefficient

As we said, the *local clustering coefficient* of a vertex x is

$$cc(x) = \frac{|\{\{y,z\}|y,z \in N(x), y \neq z, y \in N(z)\}|s}{\binom{d(x)}{2}}$$

A related notion is that of *global clustering coefficient*

$$cc_G = \frac{\sum_x cc(x)}{n},$$

the average clustering coefficient of its vertices.

► How can one efficiently compute or approximate the local/global clustering coefficient?

► Here we consider the local case

Triangles of an edge

Define, for every edge yz

$$T(yz) = |N(y) \cap N(z)|.$$

This is the number of triangles that the edge yz closes.

From this, you can define

$$T(x) = \sum_{y \in N(x)} T(xy),$$

hence

$$cc(x) = \frac{T(x)}{2\binom{d(x)}{2}}$$

because $T(x)$ counts every triangle twice...

Jaccard coefficient of an edge

The problem thus can be reduced to computing, for every edge yz,

$$T(yz) = |N(y) \cap N(z)|.$$

Recall the notion of Jaccard coefficient:

$$J(A, B) = \frac{|A \cap B|}{|A \cup B|}$$

Equivalently:

$$\frac{1}{J(A, B)} = \frac{|A \cup B|}{|A \cap B|} = \frac{|A| + |B| - |A \cap B|}{|A \cap B|} \frac{|A| + |B|}{|A \cap B|} - 1.$$

Hence

$$|A \cap B| = \frac{|A| + |B|}{1 + \frac{1}{J(A,B)}}.$$

Jaccard coefficient through min-wise permutations

So the problem is further reduced to computing, for every edge yz,

$$J(yz) = J(N(y), N(z)),$$

after which

$$T(yz) = \frac{d(y) + d(z)}{1 + \frac{1}{J(yz)}}.$$

Recall that

Theorem

Let $A, B \subseteq \Omega = \{0, 1, \ldots, M - 1\}$, and let Π be the set of all $M!$ permutations of Ω. If π is drawn uniformly at random from Π

$$P[\min(\pi(A)) = \min(\pi(B))] = J(A, B).$$

So, the idea to compute $J(N(y), N(z))$ is:

- generate a random permutation (i.e., renumbering) π of the nodes
- compute $\min \pi(N(y))$ and $\min \pi(N(z))$
- if the two values coincide, count $+1$

Repeat the above procedue many times, and use the fraction of $+1$'s to estimate $J(N(y), N(z))$.

Some further notes:

- we have a counter per edge $C[yz]$ (to count the number of $+1$'s): we keep them on external memory
- to know if $\min \pi(N(y)) = \min \pi(N(z))$ we must have computed the minima before: we need two passes
  - first pass: generate the permutation π and compute the minima $\min \pi N(-)$) (kept in central memory)
  - second pass: increment the counters
- we use hashing instead of permutations (equivalent, provided that the probability of collision is negligible).

The algorithm (1)

```
for K times do
    generate a hash function h : V_G → N
    for each x ∈ V_G do
        M[x] ← min_{y∈N(x)} h(y)
    end for
    for each x ∈ V_G do
        for each y ∈ N(x) do
            read C[xy] from disk
            if M[x] = M[y] then
                C[xy] ← C[xy] + 1
            end if
            write C[xy] to disk
        end for
    end for
end for
```

for each $x \in V_G$ **do**
   $T(x) \leftarrow 0$
   for each $y \in N(x)$ **do**
     read $C[xy]$ from disk
     $T(xy) \leftarrow (d(x) + d(y))/(1 + K/C[xy])$
     $T(x) \leftarrow T(xy)$
   end for
   $cc(x) \leftarrow T(x)/(d(x)^2 - d(x))$
end for