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Measurement and Approximation of Infinite Structures via Tolerance Spaces

— Preliminary Version —

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Chapter 1

Preface

1.1 Structure of the thesis

For convenience of the reader, we start with a general overview of the content of this thesis.

In Chapter 2, we motivate the notion of tolerance space by using some examples coming from the theory of computation, measurement, aynchronous circuit design and from the relativistic theory of time (with applications to concurrency theory).

After introducing some definitions and theorems from domain theory (Chapter 3), we discuss at a certain depth the rôle of tolerance spaces (and their generalizations) for obtaining results in the theory of domains, with applications to the construction of universal domains and to the solution of recursive domain equations (Chapter 4). In this context, we present some generalizations of Rado's theorem about the existence of a universal tolerance space (Section 4.3), and give, in Section 4.4, some hints about how these constructions lead directly to the possibility of solving recursive domain equations by using suitable number-theoretical encodings. We also consider two alternative universal constructions, one based on trace automata and prime event structures (Section 4.5), and one of probabilistic flavour (Section 4.6).

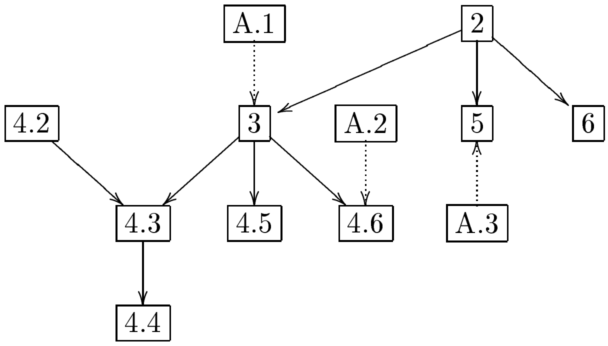
In Chapter 5, we see how one could endow a tolerance space with a further topological structure, and use this fact to obtain finitary approximations of complex topological spaces.

Finally, Chapter 6 takes into thorough consideration those tolerance spaces which could be adopted for representing results of measurements, concentrating on those associated with strongly noetherian semiorders, and obtaining some simple characterizations of tolerance properties in terms of the structure of maximal chains and antichains in the associated semiorder.

In the Appendix, we recall the basic notions of category theory, measurement theory and general topology which the reader should possess in order to understand the contents of this thesis.

Most results presented in Chapter 3 are standard, except some which were published in [BCS93]. The material of Chapter 4 is mostly new, except for Session 4.5, which appeared in [BCS93]. Some of the results presented in Chapter 5 were part of [Bol95]. Most of the content of Chapter 6 already appeared in [Bol96].

Here follows the precedence diagram of the thesis.



1.2 Notations, conventions and terminology

In the sequel, we shall freely use some very standard notations and conventions. Here is a table of some less common notations; some more are introduced in the Appendix.

Notation	Meaning
$A \subseteq_{\text{fin}} B$	A is a finite subset of B
$A \subset B$	A is a proper subset of B
A^C	the complement of A
$A \setminus B$	A minus B (set-theoretic difference)
$\prod_{i \in I} A_i$	the cartesian product of the family A_i
$\wp(A)$	the powerset of A (i.e., the set of all subsets of A)
$\wp_{\text{fin}}(A)$	the set of all finite subsets of A
$f \circ g$	the function composition of $g : A \to B$ and $f : B \to C$
$x \, R \, y$	the pair (x, y) belongs to the relation R (i.e., $(x, y) \in R$)
$R \circ T$	the composition of two relations $R \subseteq A \times B$ and $T \subseteq B \times C$
$\phi \iff \psi$, ϕ iff ψ	ϕ if and only if ψ
$\phi \wedge \psi$ ($\phi \vee \psi$, $\neg\phi$)	ϕ and ψ (ϕ or ψ, not ϕ)
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$	the sets of natural, integer, rational, real numbers, resp.
ω	the set of natural numbers; the first infinite ordinal

In order not to puzzle the reader, a short warning about the terminology used in this thesis is needed. In the following, we shall mainly be concerned with tolerance spaces, and with their applications to domain theory, approximation theory and measurement. As a matter of fact, from a definitional point of view, a tolerance space is nothing more than a reflexive (possibly infinite) graph, and we could have chosen to use the standard graph-theoretical terminology to speak about its properties. Nevertheless, we feel that the idea of tolerance space has a peculiar significance of its own which should be always emphasized, if only because of the way in which the adjacency (tolerance) relation is generated.

For this reason, we have decided to adopt a completely tolerance-theoretic nomenclature, and to adhere to this general principle throughout the whole thesis, with few exceptions. Much for the same reasons, we have decided to use a domain-theoretical terminology which fits our needs, thus discarding some of the possible (sometimes, widely diffused) alternatives.

We give here a brief list of the alternative names the reader may find in the literature, which will be anyway mentioned in the text whenever introducing a new concept.

Name adopted here	Possible alternatives
tolerance space [Zee62]	(undirected reflexive) graph
tolerance-continuous function	graph morphism
(tolerance space) embedding	graph embedding
indifference chain	path in a graph
atomic coherent dI-domain	coherence space [Gir87]
atomic dI-domain	qualitative domain [Gir86]
line (cut) [Pet96]	maximal (anti)chain

1.3 Acknowledgements

There are no words to express the gratitude I owe to all those people who, in one way or another, have helped me in the writing of this thesis. My first and biggest "thank" goes to Felice Cardone, who constantly helped me since my graduation, and followed my research activity with ardour, zeal and competence. Nicoletta Sabadini supported my work with patience and trust. Manfred Droste had a fundamental rôle in the development of many ideas and results, and gave me unvaluably useful advices of both technical and motivational nature.

Many other people contributed with comments, advices and discussions to the making of this thesis: among them, I would like to thank Sebastiano Vigna (who helped me for the categorical part, and read some drafts) and C.A. Petri (who made useful comments on a preliminary version of my paper [Bol96]).

But, more importantly, I would like to dedicate this thesis to my parents: their constant help, precious advices, and their silent but everlasting presence contributed to this work, and to my life, in a way which words can hardly (if ever) say.

Chapter 2

Introduction: tolerance and approximation

Concurrency is a basic computational phenomenon occurring in spatially distributed systems in which communication between components takes a non-negligible time.[1] Its study has gained much impetus from the insight that the *processes* of such systems can be analyzed as sets of *events* (or, better, event occurrences) with an order relation on them reflecting the order in which such events may become enabled in a run of the system.[2] Two events are concurrent exactly when they are incomparable in this ordering, meaning in particular that none of them is enabled by the other (early instances of this view can be found in Holt et al. [H$^+$68] and Patil [Pat70]).

The source of this basic insight can in fact be traced back to the work of Carl Adam Petri [Pet62, Pet82a, Pet77] who later (see for example [Pet79, Pet80a, Pet82b, Pet87, PS87]) developed an axiomatic *theory of concurrency* inspired by earlier axiomatizations of relativistic physics (as summarized, for example, in Carnap [Car58]). The basic framework for studying the concurrency relation is a structure

$$(X, \mathrm{co}) \tag{2.1}$$

for an arbitrary set $X \neq \emptyset$ with a reflexive, symmetric binary relation co over X. This is what in the present work will be called a *tolerance space*, co being a *tolerance relation*, following the terminology introduced by Zeeman [Zee62].

Of course this axiomatic development favors the comparison of properties of concurrency with those of formally similar structures arising in apparently unrelated areas. We shall now comment briefly on some of these connections and pointers to the relevant literature, leaving to later sections a closer study of some of them.

Petri himself (in Petri [Pet80a]) explicitly observed the close link between axioms for structures of the form (2.1) and properties of the relation of *indifference* which arises in ordinal measurement of utility. This can be extended to an interpretation in which

[1] This is the definition of a distributed system given, for example, in Lamport [Lam78]. Observe that any system is distributed in this sense, when looked at sufficiently closely.

[2] This is usually called *causal order* in the literature, although it represents the same kind of relation which exists between cocking the hammer of a gun and pulling the trigger, which can hardly be counted as one of cause and effect.

X is a set of samples to be measured by some device M. Then $x \operatorname{co} y$ if x and y are *indistinguishable* by means of M. It was Luce [Luc56], on the basis of previous works in economics [Arm39] and philosophy [Goo77], who supported the claim that indifference should be regarded as a tolerance relation, formalized as incomparability in partial orders of a well-behaved class. These are the *semiorders* which, by results of Scott and Suppes [SS58], can be represented as the orderings arising when the measuring device outputs for each sample a real value within a degree of precision limited by a threshold depending on the device only.[3]

Intransitive relations of similarity appear quite naturally in the analysis of what Poincaré [Poi03] called *empirical continua*. Indeed, the existence of sensory thresholds (or of physical limitations on measuring devices) suggests that density properties of orderings arising from the classification of perceptual data be replaced by just such a relation.[4] Poincaré even turned the main consequence of these limitations into a definition, which is also taken to be a fundamental property of concurrency in Petri [Pet80a]:

> On a observé, par exemple, qu'un poids A de 10 grammes et un poids B de 11 grammes produisaient des sensations identiques, que le poids B ne pouvait non plus être discerné d'un poids C de 12 grammes, mais que l'on distinguait facilement le poids A du poids C. Les résultats bruts de l'expérience peuvent donc s'exprimer per les relations suivantes:
>
> $$A = B, B = C, A < C$$
>
> qui peuvent être regardées comme la formule du continu physique. [...] Un système d'éléments formera un *continu*, si l'on peut passer d'un quelconque d'entre eux à un autre également quelconque, par une série d'éléments consécutifs tels que chacun d'eux ne puisse se discerner du précédent. Cette *série linéaire* est à la *ligne* du matématicien ce qu'un *élément* isolé était au point. (*ibidem*, pages 34–35 and 45; English translation pages 22 and 31)

This line of thought has been further pursued by Zeeman [Zee62, ZB70] motivated by applications to biology and physics, and by Poston [Pos71], who reconstructs a substantial amount of results from higher mathematics replacing topological notions by their finitistic analogues in the context of tolerance spaces of the kind (2.1).

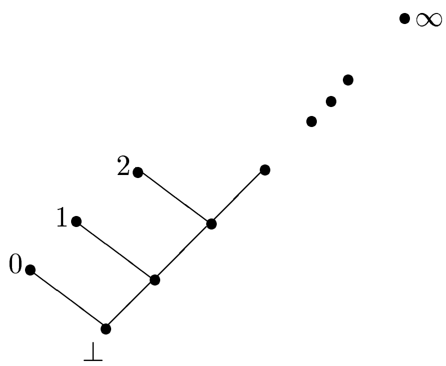
A different connection appears when X is taken to be the collection of convex sets (intervals) of an ordered set; in this case co may be interpreted as overlapping of intervals, and further axioms may be imposed depending on the order-theoretical and topological properties of the underlying ordered set. This interpretation originated with Whitehead's method of "extensive abstraction" (see Wiener [Wie14, Wie16] for early applications), and is especially relevant to what has come to be known as "pointless topology," in which the points of a topological space are not primitive entities but rather are constructed as suitable collections of open sets. On the one hand, this view conforms to the fact

[3] In Chapter 6 of the present work we take up the analysis of a special class of semiorders which is of interest both in measurement and in the theory of concurrent systems.

[4] It should be observed that limitations in this case have a positive counterpart which appears when we realize that the very existence of a practice of measurement in everyday life is closely dependent on the fact that, for example, sameness of weight is a tolerance relation which is not an equivalence—otherwise we should be able to communicate real numbers with infinite precision.

that the infinite precision in measurement needed to exhibit a point can be obtained only in the limit. In one such approach, for example, points are identified with maximal sets of pairwise overlapping elements of X, as in the theory developed by Wallman and summarized in Menger [Men40]. On the other hand, it has recently turned out that the uniform application of this standpoint allows to prove constructively results which classically need some form of the axiom of choice (for an example, see Coquand [Coq91, Coq92]).

On the logical side, the elements of X can be considered as atomic propositions, co describing a relation of compatibility or consistency between propositions. It is natural in this context to take sets of pairwise compatible propositions as (partial) elements of some kind of Scott domain.[5] The idea is due to Girard and is developed in Girard, Lafont and Taylor [GLT89]. For example, the domain of "lazy natural numbers" which is of interest for the denotational semantics of lazy functional programming languages and whose Hasse diagram has the shape:



is easily seen to be described by the set of propositions:

$$X = \{\boldsymbol{n} \mid n \in \omega\} \cup \{\boldsymbol{n}^+ \mid n \in \omega\}$$

with consistency given by the (symmetric closure of the) pairs:

- $\langle \boldsymbol{m}, \boldsymbol{n} \rangle$ for $m = n$,
- $\langle \boldsymbol{m}^+, \boldsymbol{n} \rangle$ iff $m < n$,
- $\langle \boldsymbol{m}^+, \boldsymbol{n}^+ \rangle$ for all m, n.

It is thus possible also in this case to connect the idea of a tolerance space, in this last interpretation, to the idea of partial element approximating a total one which underlies much of Scott's work in domain theory (see especially Scott [Sco70]). This is a different way of relating domains and concurrency from that which motivated the introduction of event structures and related classes of partial orders (Nielsen, Plotkin and Winskel [NPW81], Winskel [Win80]).

[5] This interpretation will be explored thoroughly in Chapters 3 and 4 of the present work.

2.1 Approximation and equivalence: a computational viewpoint

The need for approximation is commonly determined by the impossibility (or the difficulty) of achieving (deciding) equivalence between objects. For example, in the practice of measurement, we say that a certain metal rod is *approximately* 1 meter long meaning either that we have no means for deciding whether its length is *exactly* 1 meter (since the measuring instruments at our disposal are not precise enough to make a non-approximated judgement), or because in any case an exact measurement would be too expensive, or even useless, in the given context.

Impossibility of achieving equivalence is sometimes determined by some limitations in the availability of resources, but sometimes it is intrinsic in the problem we are considering. In this section, we shall begin presenting some computational problems in which these situations take place, and show how the difficulties arising may be well explained in the context of observability.

We describe now a simple *Gedankenexperiment* which provides an example of a situation in which exact judgements about an equivalence relation require an unbounded amount of effort (in this case, computational effort).

Consider a set $A \subseteq \omega$ of natural numbers; clearly, this induces an equivalence relation on ω defined by

$$x \sim y \text{ if and only if } (x \in A \iff y \in A).$$

In other words, two numbers are equivalent iff they are both contained in A, or both contained in its complement. This equivalence relation has exactly two equivalence classes, A and A^C (the complement of A). Naturally, there is a direct relation between decidability of A and that of $\sim$: more precisely, $\sim$ is decidable if and only if A is recursive.

In fact, suppose that A is recursive, and let M be a deterministic Turing acceptor which decides the membership problem for A; in other words, M is a machine which terminates for every input in one of two possible states YES or NO, and $M(x) =$ YES if and only if $x \in A$. Now, to decide whether $x \sim y$ or not, simply run M on x and y, and answer "YES" if and only if M gives the same answer on both inputs (i.e., $M(x) = M(y)$).

The problem here is that, in general, we do not know in advance how long it will take for the machine M to give an answer: we only know that the machine will eventually halt, but the number of steps is in general unbounded. Suppose that we need to take some decision (possibly: an approximate one) within a fixed number t of steps, using the machine M. What should we do? We could run M, and hope that it halts on the given input *within* t steps: if we are lucky enough, the machine will stop within the required bound, and we have the answer we needed. But, what if the machine has not yet ended its computation after the t-th step? The only thing we can do is to output some kind of "don't know" answer, expressing the fact that M has not been able to compute an answer to the problem in that fixed amount of time.

In other words, we build a new machine M_t which acts as follows:

$$M_t(x) = \begin{cases} M(x) & \text{if } M \text{ halts within } t \text{ steps on input } x \\ \text{HK} & \text{otherwise.} \end{cases}$$

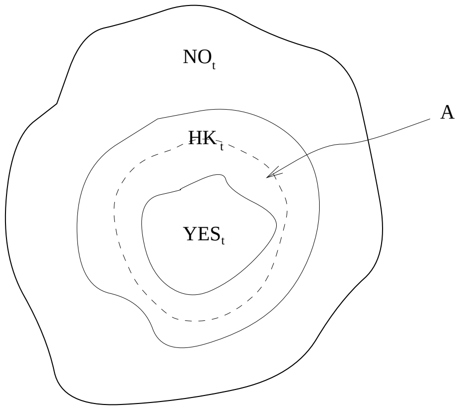


Figure 2.1: Partition induced by M_t

The machine M_t actually uses at most t steps, but it sometimes gives a "don't know" answer (represented by HK, i.e., "Heaven Knows"); note that M_t is *not* an acceptor. Rather, it classifies the set of inputs ω into three disjoint subsets YES_t, NO_t and HK_t, where $\mathrm{YES}_t \subseteq A$, $\mathrm{NO}_t \subseteq A^C$ and HK_t is the set of all instances for which no solution (acceptance or rejection) was found within t steps; the situation is sketched in Fig. 2.1.

In a sense, M_t is a degraded version of M, and it "recognizes" an approximated version of the set A. It is natural to ask what kind of "equivalence" relation could be induced by using M_t instead of M. What we want to have is a kind of approximation of $\sim$, but how could we define it? Clearly, if M_t gives an exact answer for both inputs (i.e., if it outputs a YES or a NO), then we can decide $\sim$ on those inputs in an exact way. But what shall we do if we get a "don't know" answer for one (or both) inputs? We may want to decide in some random way (for example, by tossing a coin), but we prefer to do this in a deterministic manner.

Of course, whatever protocol we choose, it shall be prone to error: the only thing we can do is to decide if we want to have a surplus of "YES" or of "NO" answers. Our choice will be to make a judgement which is never wrong when it answers "NO", but may be wrong when it answer "YES"[6]. In practice, we can define a relation $\sim_t$ by putting

$$x \sim_t y \text{ if and only if } M_t(x) = M_t(y) \text{ or } M_t(x) = \mathrm{HK} \text{ or } M_t(y) = \mathrm{HK}.$$

In other words, we answer "YES" whenever there is some chance that "YES" is the right answer, while answering "NO" only if we are sure that "NO" is correct.

Now, what kind of relation is $\sim_t$? It is certainly reflexive and symmetric, but it is *not* transitive; in fact, suppose that x is accepted within t steps, z is rejected within t steps, and y is neither accepted nor rejected during that period. As far as we know, x *could be* equivalent to y, and likewise y could be equivalent to z, but x is certainly *not equivalent* to z: i.e., $x \sim_t y$ and $y \sim_t z$ but $x \not\sim_t z$.

[6]This is in accordance with what happens in measuring: if we are to compare, for instance, two objects by means of an arm balance, we can safely consider the objects to have different weights whenever this is the response of the scale, but when the arms are in equilibrium, we are allowed to consider the objects to be indistinguishable (as for weight) only inasfar we cannot use a more precise scale (thus putting more "effort" in the decision).

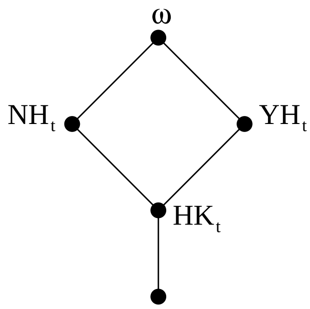


Figure 2.2: The topology Ω_t of observable properties

As a matter of fact, the graph of $\sim_t$ is the dual of a complete bipartite graph with some isolated node added: more precisely, we have the three cliques YES_t, NO_t and HK_t, plus all the arcs connecting elements of YES_t to elements of HK_t, and elements of HK_t to elements of NO_t. Said otherwise, $\sim_t$ is the complete relation minus $(\mathrm{YES}_t \times \mathrm{NO}_t) \cup (\mathrm{NO}_t \times \mathrm{YES}_t)$. Note that $\sim_t \supseteq \sim_{t+1}$ and, moreover, $\cap_{t\in\omega} \sim_t = \sim$, because $\mathrm{YES}_t \subseteq \mathrm{YES}_{t+1} \subseteq A$, $\mathrm{NO}_t \subseteq \mathrm{NO}_{t+1} \subseteq A^C$ and $\mathrm{HK}_t \supseteq \mathrm{HK}_{t+1}$ (with $\cap_{t\in\omega}\mathrm{HK}_t = \emptyset$).

Thus, $\sim_t$ is not an equivalence relation, because it is not transitive, but it "approximates" an equivalence relation, in a sense which will be made precise in Chapter 5: relations which are reflexive and symmetric, but not transitive, like $\sim_t$, are usually called "tolerance relations" [Zee62]. What we want to discuss here is just that the problem of achieving equivalence is related to the strong bound we have imposed on the resources (in this case: time) we can use.

This problem can be clearly rephrased in terms of "observability"; what kind of properties can we observe at time t? If the word *property* is intended in its extensional sense as a "set of objects" (a set of numbers, in this case), then the properties we can observe are:

- the trivial properties $\emptyset$ and ω;
- the property of "possibly being in the set A", which is represented by $\mathrm{YH}_t = \mathrm{YES}_t \cup \mathrm{HK}_t$;
- the property of "possibly being out of the set A", which is represented by $\mathrm{NH}_t = \mathrm{NO}_t \cup \mathrm{HK}_t$.

This defines a *topology of observable properties at t*, say Ω_t, and a base for the open sets of Ω_t is given by $\{\mathrm{YH}_t, \mathrm{NH}_t\}$; the open sets of Ω_t are represented in Fig. 2.2. (The idea that observable properties form a topology is well-motivated and explained in [Smy92].)

Now, this clearly induces a topology on the limit, i.e., the topology $\Omega = \cup_{t\in\omega}\Omega_t$, whose open sets are just those which are observable at some finite step. We can quotient the space (ω, Ω) by the relation $\sim$, thus obtaining a topology Ω' on the two-element set $\{A, A^C\}$: which topology do we obtain? Just observe that $\{A\}$ is open if and only if $\{A\} \in \Omega_t$ for some t, i.e., if and only if M works in constant time, and in this case also $\{A^C\} \in \Omega_t$. So, the topology Ω' is either indescrete or discrete (i.e., the Sierpinski topology is ruled out). Moreover, Ω' is discrete if and only if M works in constant time.

This idea of inducing a topology on the limit starting from topologies of finite observations, will be further pursued in Chapter 5, where we shall also prove that it can be used to give finite approximations for uncountable topological spaces, like the Euclidean space.

In the above example, equivalence could actually be decided if we only imposed no bounds on the response time. We shall now present a situation in which equivalence is undecidable: there is no way to decide whether two objects are equal, not even using unbouded resources. Once more, the problem here is closely related to the problem of making finitely-decidable observations. The following example is borrowed from [Smy92], with minor changes.

Consider a device which outputs an infinite binary sequence, one bit at a time: the set of all possible outputs is thus $\{0,1\}^\omega$ (the set of all infinite sequences of bits). Each such sequence $\mathbf{x} = \langle x_0, x_1, \ldots\rangle$ can be interpreted as the real number, denoted by x, whose binary representation is $.x_0x_1x_2\ldots$. Thus, for example, the sequence $\langle 0,1,1,0,0,0,\ldots\rangle$ is taken to represent the number $3/8$.

An observer inspects the output sequence as it proceeds, noting various properties of it. Since the device is a "black box", his judgements can only be based on the finite segments which have been output so far. A preliminary question is what kind of properties (i.e., subsets of $\{0,1\}^\omega$) are observable?

Of course, a property is observable if and only if it can be observed within certain (finite) time; in other words, the only basic observable properties are those of the form $w\uparrow$ where w is a finite binary string, and $w\uparrow$ denotes the set of all infinite sequences having w as prefix. For example, the property of "starting with a 0" is observable, while the property of "containing finitely many 0's" is not.

Notice that, if P and P' are observable properties, then also $P \cap P'$ is such; moreover, an arbitrary disjunction of observable properties is also observable[7]. Thus, the observable properties form a topology Ω_{obs} on $\{0,1\}^\omega$, and $\{w\uparrow : w \in \{0,1\}^*\}$ is a base for this topology.

Smyth [Smy92] discusses the topology Ω_{obs} in depth[8]; what we need, for our present purpose, are just some very simple properties of this topology.

Suppose you possess two devices like that introduced above, each outputting some (unknown) sequence; let $\mathbf{x}$ and $\mathbf{y}$ be the two sequences. We want to decide whether $x = y$ or not. Is this an observable property? Of course not. Even if, at a certain point, we have observed the same finite prefix of both strings, we cannot say that the two (infinite) sequences represent the same value (not even when they are actually "the same" sequence), because we cannot stake claims on the future.

Yet, even though the problem $x = y$ is undecidable, we can still try to approximate its solution, much in the same way as we did for $\sim$ above. For example, we could say that $x =_t y$ if the two prefixes $w = \langle x_0, \ldots, x_t\rangle$ and $v = \langle y_0, \ldots, y_t\rangle$ observed at the t-th stage

[7] This is a simplification: the view that observable properties have the same closure properties as the open sets of a topological space has been advocated by Abramsky in [Abr87]. For a criticism and sharpening of this view, compare the discussion in [Smy92].

[8] Actually, Ω_{obs} can be obtained in a straightforward way; just order $\{0,1\}^\infty$ (the set of finite and infinite strings on $\{0,1\}$) by prefix, and induce the Scott topology on it. Then Ω_{obs} is just the subspace of maximal elements; it is actually homeomorphic to the Cantor space (Section 5.7; see also [Smy92]).

are compatible, in the sense that there are at least two possible elements of $w\uparrow$ and $v\uparrow$ representing the same real.

Once more, the relation $=_t$ is not an equivalence relation: there are some pairs which are certainly ruled out as non-equivalent, but there may be some other pairs which are considered as "possibly equal" only because there is some possibility that they turn out (in the future) to be equal (i.e., to represent the same real number). For example, the two sequences $\langle 1,0,0,0,0\rangle$ and $\langle 0,1,1,1,1\rangle$ must be considered equal, because they can be prefix of $\langle 1,0,0,0,0,\ldots\rangle$ and $\langle 0,1,1,1,1,\ldots\rangle$, both representing the number 1/2.

Also in this case, $=_t \supseteq =_{t+1}$ and moreover $=$ is the intersection (limit) of the $=_t$'s. So, our tolerance relations *do approximate* the (undecidable) equivalence $=$.

2.2 Two case studies

In the present section, we study more extensively two examples of situations in which tolerance relations arise, and which inspired, in different ways, the first studies of the abstract relation of concurrency.

Given the definition of a distributed system as one in which transmission delays are not negligible, it is natural to consider the components of such systems as asynchronous devices: in fact, for a large system, there is no guarantee that the clock period needed for achieving synchrony of the components is wide enough for it to be perceived correctly throughout the system. The first example is therefore a case study which relates the idea of tolerance with some issues in asynchronous circuit design. Much of our discussion is based on [Sei80], where a thorough analysis of self-timed asynchronous systems is presented (see also [Kat94]).

On the other hand, we have already mentioned the fact that Petri's original axiom system for concurrency was inspired by the early attempts to axiomatize relativistic physics; the reason is that relativistic simultaneity is a tolerance relation which cannot be an equivalence. Thus, we also discuss in some length some basic facts about relativistic simultaneity in a model-theoretic setting.

2.2.1 Using tolerance as a design tool: the case of asynchronous circuits

Much of the design of a system is concerned with functional aspects which can be described in a metric-free topological setting, with the help of logic diagrams, circuit diagrams or such, which allow the designers to concentrate on the system behaviour at a level of abstraction where implementation details are immaterial. Nevertheless, sooner or later, it becomes necessary for the designer to think about the spatial geometry of the system, which is governed by specific physical laws, determining also the behaviour of the circuit in time.

This requires the use of some kind of discipline to establish a set of signalling conventions on the system interconnections and element timing, in order to obtain the correct sequencing of events happening at different locations of the system, and to prevent from inconsistent behaviours.

There are basically two approaches one can use for defining such disciplines: synchronous systems and self-timed (asynchronous) systems. In the former case, sequencing and time are connected by means of a global clock signal, which synchronizes events throughout the whole system. In the latter, the connection between sequence and time is maintained only locally, in the interior of the various atomic parts of the system (the so-called "elements"), while some kind of signalling protocol is used to maintain global consistency.

Even though synchronous systems are by far the most widely used at present, they present some (at least, potential) serious limitation: some of these are related to the difficulties of moving information from point to point within a single global-clock period, and of managing very large designs in a framework in which all system parts must operate together in "lockstep". Moreover, if a system is made of many independently-timed parts, the problem of clock synchronization becomes a major issue; unfortunately, clock synchronization cannot be accomplished with complete reliability, due to the presence of metastable states [Sei80].

Therefore, the self-timed discipline seems the most promising: each element can be designed simply as a synchronous system, with the possibility of stopping and restarting the local clock at any time. Signalling conventions are then used to synchronize the various parts in a delay-insensitive fashion (i.e., in a way which abstracts from communication delays).

In this subsection, we shall first present some examples of signalling protocols which are typically used in the design of self-timed systems; then, we shall further discuss some implicit assumptions which are usually considered when using the self-timed approach, introducing the notion of "equipotential region". Finally, we shall see how this is related to some idea of observability which can be described by using tolerance-continuous functions.

Two-cycle vs. four-cycle signalling

Consider an asynchronous system with two agents which must interact: one agent (called *master*) is to request the other (called *slave*) to process a certain set of data, and the process should proceed with the slave providing an output back to the master. We assume that there is no global clock for synchronizing the communication between master and slave, which amounts to saying that the communication process requires a non-negligible delay, and the protocol must guarantee correctness regardless of the delays. Such protocols, or signalling conventions, are usually termed *delay-insensitive*, and are typical in every approach to the design of asynchronous systems.

There are two kinds of communication lines from the master to the slave: the first is represented by the input data **I** to the slave, and the second by the signal lines needed to perform the communication protocol. Also, two kinds of lines exit from the slave, i.e., the output lines **O** and the lines providing feedback signals to the master.

The easiest way to implement the communication protocol is the following. Suppose that there are only two signal lines, one from **M** (the master) to **S** (the slave), called the request line (**req**), and one from **S** to **M**, called the acknowledge line (**ack**), both of them carrying binary information (so, they can be either low or high). Initially, both **req** and **ack** are low (in symbols, **req** $\downarrow$, **ack** $\downarrow$). The protocol proceeds as follows: when **M** has prepared the input, it asserts **req** (**req** $\uparrow$) and leaves the input stable (untouched), in order to allow **S** to read it correctly. When **S** notices that **req** is high, it processes the input, prepares the output, and then signals back to **M** by asserting **ack**, which so becomes high. When **M** notices that **ack** is high, it can read the output.

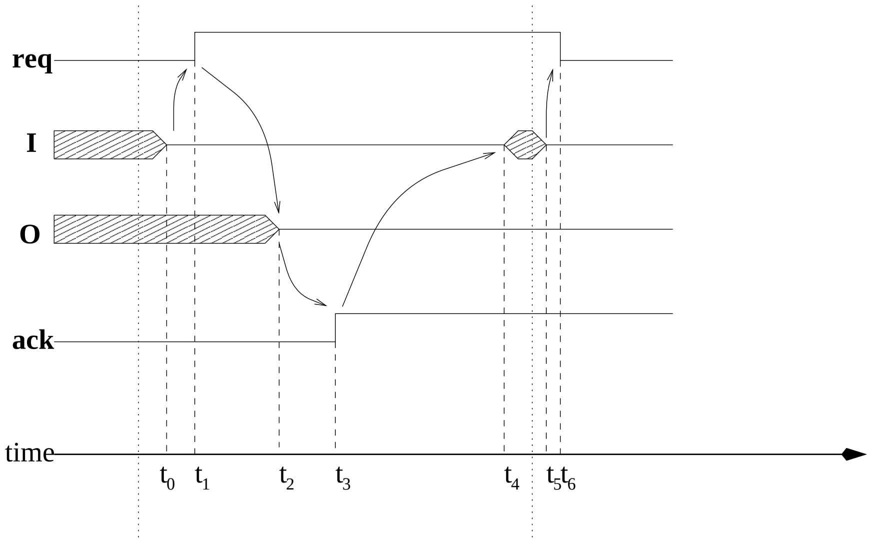


Figure 2.3: Two-cycle signalling (space-time diagram)

This protocol is known as *two-cycle signalling*, and could be depicted in a waveform space-time diagram as shown in Fig. 2.3.

We interpret the diagram as follows: **req** and **ack** are depicted simply as lines which can be either low ( ) or high ( ); **I** and **O** can be either stable and correct (straight line) or not (dashed area).

For sake of simplicity, we assume to have a global (real) time axis. Initially (at time 0) **req** and **ack** are low. At time t_0, inputs are ready and the master asserts **req** (time t_1). When the slave notices this, it prepares the output (time t_2) and asserts **ack** (at time t_3). When **M** notices that **ack** is high, it reads the output and changes the input, preparing it for a new request (time t_4). Now, when input is ready (at time t_5), and it unasserts **req** (which happens at t_6), to let the slave know that a new input is to be processed.

This starts another request/acknowledge cycle (the first cycle is that appearing between the two dotted lines in the above diagram), which is similar to the previous one, but where the rôles of low and high are changed, because now the two signal lines are both high.

The master and slave algorithms can be written informally as follows:

MASTER $\equiv$ **loop**
  prepare input;
  flip **req**;
  wait **ack**;
  read output;
forever

SLAVE $\equiv$ **loop**
  wait **req**;
  process input;
  prepare output;
  flip **ack**;

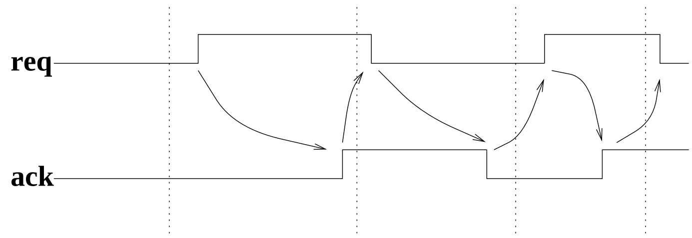


Figure 2.4: Two-cycle handshaking (space-time diagram)

forever

Here, we used the two following primitives:

- *flip* change the value on a line (if it was low, it becomes high, and viceversa);
- *wait* wait until the value on a line changes.

In the very special case of a communication where both input and output are empty, this protocol reduces to a simple handshaking protocol, the so-called *two-cycle handshaking*. In this case, the space-time diagram (which now involves only the **req** and **ack** lines) is simply that presented in Fig. 2.4 (as usual, we have used a pair of dotted lines to represent a single request/acknowledgement cycle; observe that the cycles start alternatively with either both lines high, or both low).

The problem with two-cycle signalling (or handshaking) is that it requires the master and slave to contain an additional state for "remembering" which is the current state of the request/acknowledge line, and to flip them accordingly. In practice, we need an extra storage for keeping track of the state of each line.

In order to solve this problem, one could design a more complex (from a transition viewpoint) solution, where the two lines are reset at the end of each cycle, so that master/slave have no need to keep track of the current state of the outgoing lines. This is called the return-to-zero signalling, or *four-cycle signalling*.

The waveform diagram of a four-cycle signalling is shown in Fig. 2.5.

Let us see how a single signalling cycle proceeds in this case. Initially, both lines are low. At time t_0, the input is ready, and so **M** asserts (time t_1) the request line to let the slave know that data are ready to be processed. When the slave notices that **req** is high, it reads the input, processes the data and produces an output (at time t_2), and then asserts **ack** (time t_3): now, both lines are high. As soon as the master notices that **ack** is high, it can read the output (which is currently stable), while the input becomes unstable (in the sense that the master can change the input value at will; this happens at time t_4). In practice, **ack** becoming high is the signal of "output ready". When the master has completed its usage of the output, it unasserts **req** (at t_5): only at that point, the slave is authorized to change the output (at time t_6) and unassert the **ack** line (time t_7).

Observe that both lines are low at the end of the cycle. In practice, the state of the two lines can be thought as follows:

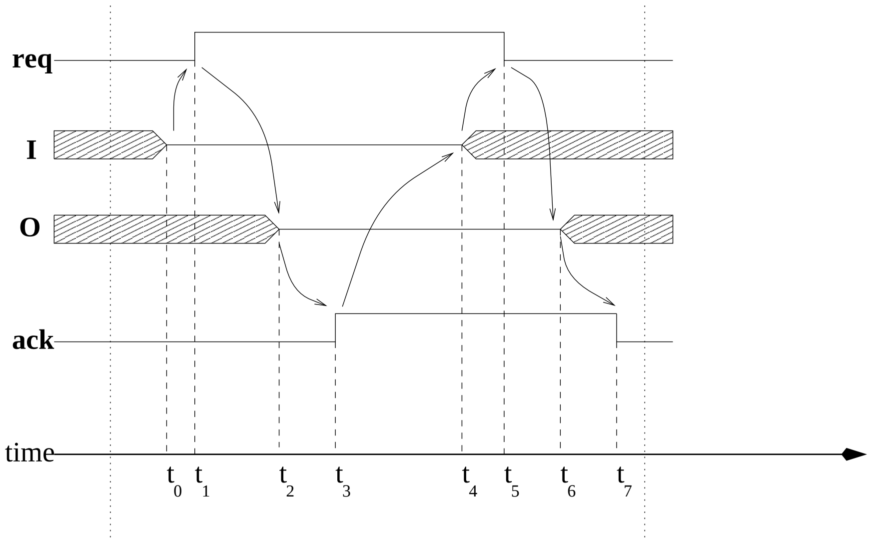


Figure 2.5: Four-cycle signalling (space-time diagram)

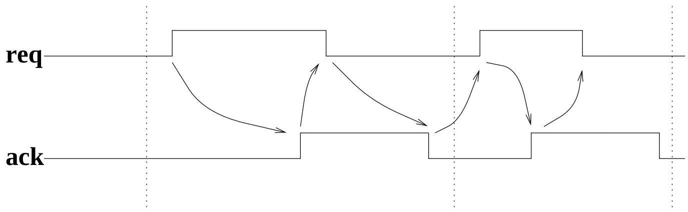


Figure 2.6: Four-cycle handshaking (space-time diagram)

- both lines are low: master has not yet submitted any request; slave is ready to process data;
- **req** is high, **ack** is low: master has prepared the input data (which are currently stable) and a request has been submitted to the slave;
- both lines are high: output from the slave is ready; in this phase, output is stable, while input may be changed;
- **req** is low, **ack** is high: master has been served, and the output may be changed; slave will become prompt to serve another request.

Notice that for the master to submit a new request, **ack** becoming low must be noticed by the master, for otherwise he would not know whether the slave has been already acknowledged that the previous output had already been read.

As before, we can ignore the data part, and sketch a simple handshaking scheme (the *four-cycle handshaking*) as in Fig. 2.6.

Observe that the algorithms corresponding to four-cycle signalling are more complicated, because they involve more transitions, but they do not make use of the *flip* primitive (which requires extra-memory):

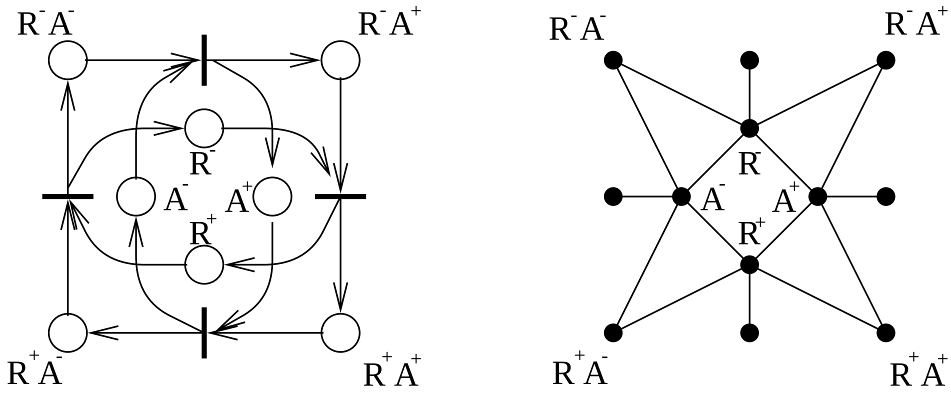


Figure 2.7: The four seasons

MASTER $\equiv$ **loop**
 prepare input;
 set **req**;
 wait-high **ack**;
 read output;
 reset **req**;
 wait-low **ack**;
forever

SLAVE $\equiv$ **loop**
 wait-high **req**;
 process input;
 prepare output;
 set **ack**;
 wait-low **req**;
 reset **ack**;
forever

Here, we used only the primitives:

- *set/reset* sets a line high/low;
- *wait-high/low* waits until a line becomes high/low.

In general, four-cycle signalling is more easy to implement than two-cycle signalling, because it does not need extra-memory for keeping track of the current state of each line.

It is tempting, at this point, to represent four-cycle handshaking in the form of the Petri net drawn in Fig. 2.7 (left), and known as *the four seasons.* The most interesting fact about this net is that it can be reconstructed from the tolerance space (concurrency relation) drawn at its right (using the techniques described by Petri in [Pet80a]), and has been indicated [PS87] as the smallest model for his axiom system for concurrency[9].

[9] It may also be interesting to observe that the partial order arising as the unfolding of this net is a semiorder which we shall encounter again in Chapter 6.

Implicit assumptions — Equipotential regions

The signalling disciplines described above are quite standard in the design of asynchronous circuits, yet we believe that a more thorough understanding of the implicit assumptions underlying such protocols is necessary.

As a matter of fact, we are dealing with an asynchronous system, which works in a self-timed fashion, and where the control is completely delegated to the single elements: there is no concept of (global) clock, but there exists a fully distributed protocol which aims at correctly sequencing the single events. A self-timed system is an interconnection of parts, which are called *elements* (in the case of our example, there are only two elements, the master and the slave): we can assume that each element has some way to preserve correctness in the sequence of events occurring locally; for example, each element may possess an internal clock which makes things happen in the right sequence.

The main problem in the design of a self-timed system is to preserve correctness in the *global* sequencing of events, even though there is no global clock to synchronize actions taking place at different elements. In order to accomplish this task, there are special signal lines which are used by the elements to communicate with one another, e.g., by indicating that a certain computation is allowed to start, or that another one has been completed. An important point, here, is that correctness of the signalling protocol must *not* depend on any assumption about delays (and, for this reason, we often speak of *delay-insensitive* protocols). In other words, the sequencing of events must be correct regardless of the delays in the communication between different elements.

One should interpret with care the necessity of preserving causal relations in the sequencing of events, since elements and connection between elements have some physical extent: according to relativistic principles, relations between occurrences of events at different points in the space may be interpreted inconsistently by observers at different locations. Moreover, if the routing and relative transmission delays are uncertain, a relation that holds in a certain physical region close to where it is created may fail to hold elsewhere. This simple observation has many consequences, and could make the discussion about properties of asynchronous systems much more complicated than is justified: in order to avoid such difficulties, one usually makes a simplifying assumption, admitting that there are small areas in the system where delays are negligible, and thus the communication can be assumed to take place instantaneously.

Such small pieces of the circuit (system) are called *equipotential regions*; as Seitz [Sei80] observes:

> This approximation is justified so long as the area is sufficiently small that the delay associated with equalizing the potential across any wire is small in comparison with switching delays or signal transition times. This approximation is roughly equivalent to an assertion that related occurrences are known to be sufficiently separated in time in comparison with wire delays that the relation will be observed to hold from any point of observation within the region.

The determination of equipotential regions is not at all obvious, and may be chosen on many different criteria; it can be characterized by defining a limit on the area, or on the

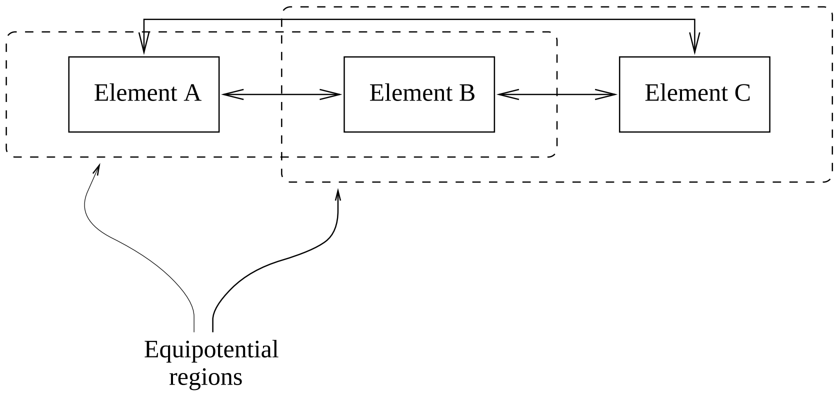


Figure 2.8: Equipotential regions defining a covering

maximal wire length within a region, and these limits will usually depend on the various layers. As Seitz notices, as far as MOS technology is considered, "a single chip is today [in 1980] a good approximation of an equipotential region [...], so long as there are no metal wires longer than about 17 mm, diffused wires longer than about 500 μ, or poly wires longer than about 300 μ".

Clearly, it is necessary for every element to be contained entirely within at least one equipotential region, but it can be included in more than one: this expresses the fact that elements are interpreted as locally synchronized parts of a system which is globally asynchronous. Equipotential regions thus do not partition the system into disjoint components, but rather define a *covering* of the set of spatial locations, each element of the covering corresponding to a single equipotential region, like in Fig. 2.8.

There is a very simple, yet meaningful, way of thinking about equipotential regions. Suppose, in a very rough view of the system, that each element is represented by a point in the Euclidean space, with a full-connection scheme (i.e., every point can communicate directly to any other), and that signal delays are proportional to the distance between sender and receiver; further, let v be the velocity of signals. The time required for a signal sent by x to reach its destination y is thus $d(x,y)/v$ where $d(x,y)$ denotes the distance between x and y. Seen in another way, the set of points to which x can signal within time t is simply the set of all points whose distance from x is at most $v \cdot t$. This defines a "forward" cone from each space-time point (called *posterior cone* in [Car58]): in other words, for every space-time point P there exists a "cone" of space-time points in the future to which P can signal. Of course, there is also an entire region of space-time in the past which can signal to P (called the "backward cone", or *prior cone*). This situation is pictured in Fig. 2.9, where the backward and forward cones of a point in a one-dimensional space are represented (the horizontal line corresponding to a time-slice, i.e., to a set of space-time points whose time coordinate is constant); a similar picture is presented in [Ben91] (Chapter I.2, Fig. 5).

Assuming the presence of equipotential regions is equivalent to assuming that signals propagate simultaneously within each region, as soon as they enter it. Fig. 2.10 shows the forward signalling cone relative to a point in the one-dimensional case, when a structure of equipotential regions is assumed.

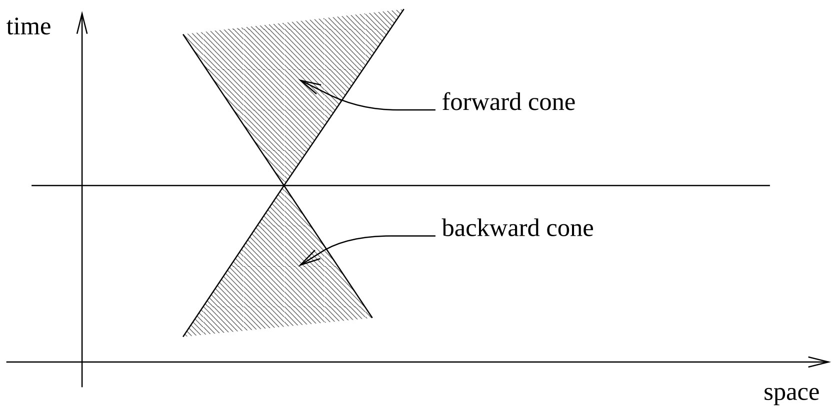


Figure 2.9: Signalling cones in the one-dimensional case

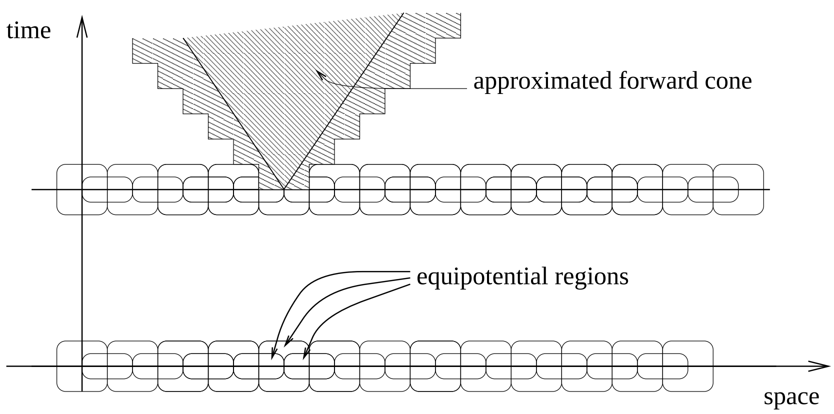


Figure 2.10: Approximated forward cone when equipotential regions are assumed

Now, we want to attack the problem of defining in a more formal way the concept of equipotential region, and provide a notion of "observability" which is suitable for a description of an asynchronous system as discussed above. One primitive idea which we certainly need is the concept of *space location*; we let L be the set of locations, which are simply the (unextended, idealized) entities where computation takes place in some unspecified way. The set of locations is covered by *equipotential regions*; in other words, we have a fixed set $\mathcal{R} \subseteq \wp(L)$ of regions satisfying the following constraints:

- every region $R \in \mathcal{R}$ is a non-empty (possibly infinite) set of locations;
- every location belongs to some region; i.e., for all $l \in L$ there exists some $R \in \mathcal{R}$ such that $l \in R$;
- if R, R' are two regions, and $R \subseteq R'$, then $R = R'$.

The two first requirements simply state that $\mathcal{R}$ is actually a (proper) covering of L; the last requirement means that we are only interested in "maximal" regions (i.e., equipotential regions cannot be nested).

As a matter of fact, we can make some stronger assumption about the structure of equipotential regions; indeed, as previously discussed, an equipotential region is simply a (maximal) set of points in the circuit which are not too far away from each other. In other words, a region is defined by a ball in the Euclidean metric space of a fixed (small) radius . Under this assumption, which implies that delays depend only on the physical distance between points, the set of regions may be described by using a binary relation of "vicinity".

Two locations l and l' are said to be adjacent if they are sufficiently close to each other (i.e., if their distance does not exceed the limit required for the level of approximation we are using), and we denote this fact by $l \operatorname{co} l'$. Thus the covering $\mathcal{R}$ is simply the set of maximal cliques of co (a region is a maximal set of adjacent points). Note that the set of regions is *not* a partition of L, unless co is an equivalence relation. As a matter of fact, overlapping of regions is not an accident, but a precise consequence of space-time continuity of signals: a signal can be directly forwarded only to a point which is sufficiently near in the space, which means that L must actually be connected under co (in the graph-theoretical sense), in turn implying that $\mathcal{R}$ is actually a covering and *not* a partition of the set of locations. Hence, formally, the relation co of vicinity is a symmetric, reflexive (but, in general, not transitive) relation, whose transitive closure is the complete relation $L \times L$.

Now, we turn our attention to observations. We can assume that there is a set V of views, each corresponding to a (partial) description of the state of the whole system. Each location, at each moment, will observe a certain view. In general, as a consequence of transmission delays, two different locations may possess, in the same instant, two different *incompatible* views of the state: this may happen because information flows in the system with a non-negligible delay. In order to describe this formally, we introduce a predicate Con of consistency; more precisely, $\mathrm{Con} \subseteq \wp_{\mathrm{fin}}(V)$ is a finitary predicate such that:

- every view is consistent; i.e., for all $v \in V$, we have $\{v\} \in \mathrm{Con}$;
- every subset of a consistent view is also consistent; i.e., if $A \in \mathrm{Con}$ and $B \subseteq A$, then also $B \in \mathrm{Con}$.

Another simplifying assumption is needed at this point, in order to make the discussion easier. We can assume that consistency is described by a binary compatibility relation $\sim$, which is a symmetric and reflexive relation[10]. In other words, a set is consistent if and only if it is a clique of the relation $\sim$.

We are now ready to relate the notion of vicinity to the concept of consistency (or compatibility). In practice, even though the views at a given instant may be inconsistent at different locations, consistency is required when the locations are adjacent: this is what we mean when saying that "the relations produced anywhere in one region hold everywhere [in the same region]" [Sei80]. Formally, the instantaneous observation of a system state is simply a function $o : L \to V$ assigning a view to each location in such a way that views assigned to points in the same region are consistent. In other words, we require that:

$$\forall R \in \mathcal{R}, \forall A \subseteq_{\text{fin}} o(R).\, A \in \text{Con}.$$

Note that consistency is expressed by saying that every finite subset is consistent.

Under our simplifying assumption, this is absolutely equivalent to requiring that the compatibility relation holds between views of adjacent locations, i.e.,

$$\forall l, l' \in R.\, l \,\text{co}\, l' \implies o(l) \sim o(l').$$

In fact, this happens if and only if the function $o : (L, \text{co}) \to (V, \sim)$ is a graph morphism, or (a term which we shall introduce later on) a "tolerance-continuous" function.

To summarize: we can describe the spatial geometry of the system as a tolerance space (i.e., undirected graph), where tolerance corresponds to vicinity, and use tolerance-continuous functions (i.e., graph morphisms) to describe observations, where the codomain is also a tolerance space of "views", with tolerance corresponding to compatibility. In a slogan, we can conclude by saying that "observability implies continuity", in the sense that every observable state is a continuous function.

2.2.2 Tolerance relations and axiomatizations of relativistic time

In this subsection, we shall make a short digression about the possibility of axiomatizing relativistic time, and prove how the relativistic notion of simultaneity assumes *unavoidably* the form of a tolerance relation. Our discussion is based on the axiomatization given in Chapter I.2 of [Ben91].

Precedence and simultaneity in a relativistic setting

Let us go back for a while to the representation of Fig. 2.9: in that case, we were considering a very simplified form of space-time, where space is just one-dimensional (i.e., we are working in the Minkowski two-dimensional space). Since we aim at studying a relativistic description of the physical world, we take c (the light velocity) as our signalling velocity.

[10]This assumption is far from being harmless, but later we shall see how one can get rid of it, working in a more abstract setting. In fact, we shall later introduce the notion of "generalized tolerance space", where tolerance is not taken to be a binary relation, but rather a finitary predicate satisfying exactly the same restrictions as Con. If we endowed both the set of locations and the set of views with such generalized tolerance relations, we would obtain much the same results.

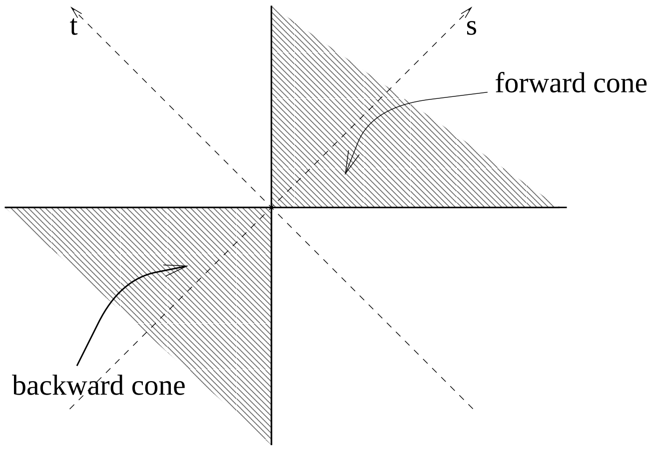


Figure 2.11: Signalling cones in the (rotated) two-dimensional Minkowski space

Thus, the forward cone of a point P in the space-time is just the set of future space-time points which can be reached from P by a signal travelling at the light velocity. A great simplification can be introduced, by assuming that $c = 1$ and by rotating the whole diagram clockwise over 45°: in this way, the forward and backward cone of the origin are simply the first and third quadrant (see Fig. 2.11).

Clearly, one can at this point introduce a precedence relation between space-time points, by postulating that

$$(x, y) < (x', y') \iff x < x' \wedge y < y'$$

(which expresses the condition under which the point (x, y) may signal to (x', y'), i.e., (x', y') is in the forward cone of (x, y)). It is worth noticing that there is no special reason to assume that space and time have the structure of $\mathbb{R}$: we can take space-time to be $\mathbb{Q} \times \mathbb{Q}$ or even $\mathbb{Z} \times \mathbb{Z}$, if we just need an approximation having countably many points. (This is in fact what one does when dealing with tools for measuring time and space having only a finite resolution power).

Now, let P and Q be any two points in the space-time; clearly, several possibilities may arise:

- P and Q may be causally related, in either direction; i.e., we might have either $P < Q$ or $Q < P$;
- it is possible that P and Q are not causally related, simply because only a signal travelling exactly at the velocity of light can connect them; we therefore define a new relation $<_c$ (which Van Benthem calls "connectability by the speed of light") by postulating that

$$P <_c Q \iff \neg(P < Q) \wedge \forall R. (Q < R \implies P < R);$$

in other words, P cannot signal to Q, but P can signal to every point to which Q can signal;

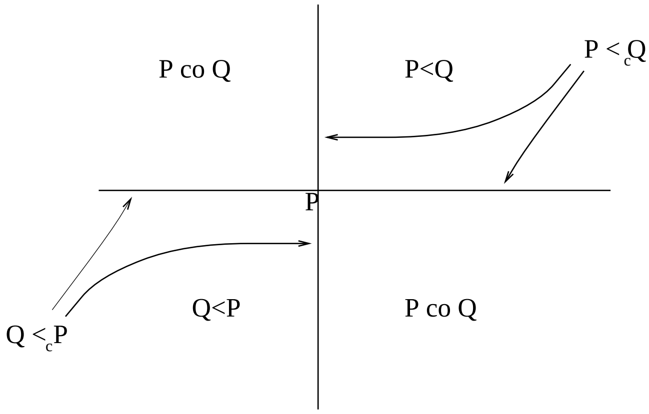


Figure 2.12: The relations of precedence, connectability by the speed of light and simultaneity

- finally, if no one of the relations $P < Q$, $Q < P$, $P <_c Q$, $Q <_c P$ holds, we say that P and Q are concurrent, or simultaneous, written $P \operatorname{co} Q$.

We present, in Fig. 2.12, how these different possibilites are realized by different points in the space-time diagram.

Now, observe that concurrency is clearly a reflexive and symmetric relation, but it is *not* transitive, i.e., it is simply a tolerance relation, and not an equivalence: this should not be surprising, after all (the same situation happens in the context of concurrency theory, where concurrency is always assumed, or turns out to be, a tolerance relation). But, is it possible to do any better? In other words, is there some way of defining simultaneity (concurrency) in a relativistic setting as to obtain an equivalence?

We shall answer negatively to the above question, by making a short digression into model theory. Observe that these observations will naturally lead to consider tolerance as the only serious candidate for simultaneity in (relativistic) physics.

First, we can assume that *precedence* in space-time is the only primitive relation, from which simultaneity should be defined by means of some logical construction. In the above discussion, we precisely defined it as follows:

$$P \operatorname{co} Q \iff \neg(P < Q) \wedge \neg(Q < P) \wedge \exists R, R'. (P < R \wedge \neg(Q < R) \wedge Q < R' \wedge \neg(P < R')).$$

In any case, simultaneity is defined "somehow" starting from precedence; but what kind of definability should we admit? We can restrict ourselves to first-order definability, or consider also higher-order logic; we quote Van Benthem's [Ben91] considerations on this subject:

> Why all this fuss about 'first-order' versus 'higher-order'? Non-logicians are inclined to think that this is a mere logician's fad. Yet this would be a mistake. The borderline between the two is a philosophically significant one, as Quine has argued repeatedly. To mention just one aspect, second-order principles are much more sensitive to the 'set-theoretic investment' made in the background

theory of our temporal structures. This shows in the independence proofs of set theory, where various 'real continua' may arise from the same Dedekind construction on the rationals, depending on the (kind of) subsets available for $\mathbb{R}$ in the set-theoretic universe. In using second-order principles we are not only discussing our temporal order, but also its super-imposed set-theoretic structure.

In the light of these statements, it seems reasonable to use only first-order logic for deriving non-primitive relations from primitive ones (and, in particular, for deriving the simultaneity relation starting from precedence).

In the following subsections, we shall prove that, in fact, it is not possible to define simultaneity (in the space-time structure) as an equivalence relation.

A short digression in model theory

Before proceeding in our discussion, a small digression in model theory is required; we are not laying any claim of precision in this discussion, and refer the reader to the specialized literature on the subject for more information (see, for example, Chapter 5 of [BM76]).
A *first-order language* $\mathcal{L}$ is defined by the following data:

- an indexed family $\langle \mathbf{R}_i \rangle_{i\in I}$ of predicate symbols, each with a fixed ariety $\lambda_i \in \omega$;
- an indexed family $\langle \mathbf{c}_j \rangle_{j\in J}$ of constant symbols;
- a countable set $\{\mathbf{v}_0, \mathbf{v}_1, \dots\}$ of variables.

An $\mathcal{L}$-*term* is either a constant symbol or a variable. The set of $\mathcal{L}$-*formulas* is then recursively defined as follows:

- atomic formulas are of the form $\mathbf{R}_i(t_1, \dots, t_{\lambda_i})$ where the t_k's are $\mathcal{L}$-terms;
- if ϕ, ψ are $\mathcal{L}$-formulas, then also $\neg\phi$, $(\phi \wedge \psi)$ and $\forall \mathbf{v}_n \phi$ are formulas.

We let $FV(\phi)$ be the set of free variables[11] of the formula ϕ; we say that ϕ is closed if $FV(\phi)$ is empty.

An $\mathcal{L}$-*structure* $\mathfrak{U}$ is defined by:

- a non-empty set U (called the "domain" of $\mathfrak{U}$);
- an indexed family $\langle R_i \rangle_{i\in I}$ of relations on U, with R_i having ariety λ_i, i.e., $R_i \subseteq U^{\lambda_i}$;
- an indexed family $\langle c_j \rangle_{j\in J}$ of elements in U (i.e., $c_j \in U$ for all $j \in J$), one for each constant.

We often say that $\mathcal{L}$ is the language of $\mathfrak{U}$. An $\mathcal{L}$-*assignment* α for the structure $\mathfrak{U}$ is a function mapping each variable $\mathbf{v}_n$ to an element $\alpha(\mathbf{v}_n)$ (or simply α_n) of U.

The *interpretation* $\mathfrak{U}^\alpha(t)$ of a term t in the structure $\mathfrak{U}$, under the assignment α, is defined as c_j if $t = \mathbf{c}_j$, and as α_k if $t = \mathbf{v}_k$.

Now, satisfaction of a formula ϕ by the structure $\mathfrak{U}$ (under the assignment α), written $\mathfrak{U} \models_\alpha \phi$, is defined inductively as follows:

[11] More formally, the set of free variables of a formula can be defined inductively as follows: the free variables of an atomic formula are exactly those variables occurring in the formula; moreover, $FV(\neg\phi) = FV(\phi)$, $FV(\phi \wedge \psi) = FV(\phi) \cup FV(\psi)$ and finally $FV(\forall \mathbf{v}_n \phi) = FV(\phi) \setminus \{\mathbf{v}_n\}$.

- $\mathfrak{U} \models_\alpha \mathbf{R}_i(t_1, \ldots, t_{\lambda_i})$ if and only if the tuple $(\mathfrak{U}^\alpha(t_1), \ldots, \mathfrak{U}^\alpha(t_{\lambda_i}))$ belongs to R_i;
- $\mathfrak{U} \models_\alpha \neg\phi$ if and only if $\mathfrak{U} \models_\alpha \phi$ does not hold;
- $\mathfrak{U} \models_\alpha (\phi \wedge \psi)$ if and only if both $\mathfrak{U} \models_\alpha \phi$ and $\mathfrak{U} \models_\alpha \psi$ hold;
- $\mathfrak{U} \models_\alpha \forall \mathbf{v}_n \phi$ if and only if, for all $u \in U$ it holds that $\mathfrak{U} \models_{\alpha[\mathbf{v}_n/u]} \phi$, where $\alpha[\mathbf{v}_n/u]$ is the assignment which is the same as α except for the variable $\mathbf{v}_n$, which is mapped to u.

It is easy to see that the validity of $\mathfrak{U} \models_\alpha \phi$ only depends on the values assumed by α on the set $FV(\phi)$. In other words, if $\mathfrak{U} \models_\alpha \phi$ holds, and if β is another assignment which coincides with α on every free variable of ϕ, then also $\mathfrak{U} \models_\beta \phi$. In view of this observation, we can use a shortcut: if ϕ contains n free variables, we shall write $\mathfrak{U} \models_{a_1,\ldots,a_n} \phi$ to mean that $\mathfrak{U} \models_\alpha \phi$ holds whenever the value assumed by α on the i-th free variable[12] of ϕ is a_i $(i = 1, \ldots, n)$. In particular, if ϕ is a closed formula, we simply write $\mathfrak{U} \models \phi$, because satisfiability for closed formulas does not depend on the assignment.

Now, every formula ϕ with n free variables defines an n-ary relation R^ϕ on U, in the following precise way:

$$(a_1, \ldots, a_n) \in R^\phi \iff \mathfrak{U} \models_{a_1,\ldots,a_n} \phi.$$

An n-ary relation R in the structure $\mathfrak{U}$ is *(first-order) definable* if there exists a formula ϕ, with n free variables, such that $R = R^\phi$.

Now, consider an n-ary relation R on a set U, and a bijection $f : U \to U$; we say that R is *invariant* under f if and only if, for all $(a_1, \ldots, a_n) \in U^n$

$$(a_1, \ldots, a_n) \in R \iff (f(a_1), \ldots, f(a_n)) \in R.$$

In particular, a *(structure) automorphism* for $\mathfrak{U}$ is a bijection f of the domain of $\mathfrak{U}$ into itself such that R_i is invariant under f (for every $i \in I$), and $f(c_j) = c_j$ for all $j \in J$. The following quite standard result of model theory will be used in the following:

Theorem 2.2.1 *Let $\mathfrak{U}$ be a structure, R an n-ary relation which is definable in $\mathfrak{U}$, and f an automorphism of $\mathfrak{U}$. Then R is invariant under f.*

Proof: Since R is definable, there will be a formula ϕ (with n free variables) such that $R = R^\phi$. We proceed by induction on the structure of ϕ.

- Suppose that $\phi = \mathbf{R}_i(t_1, \ldots, t_m)$. Then:

$$\begin{aligned}
&(a_1, \ldots, a_n) \in R^\phi \iff \mathfrak{U} \models_{a_1,\ldots,a_n} \mathbf{R}_i(t_1, \ldots, t_m) \\
&\iff (\mathfrak{U}^{a_1,\ldots,a_n}(t_1), \ldots, \mathfrak{U}^{a_1,\ldots,a_n}(t_m)) \in R_i \\
&\iff (R_i \text{ being invariant under } f)\ (\mathfrak{U}^{f(a_1),\ldots,f(a_n)}(t_1), \ldots, \mathfrak{U}^{f(a_1),\ldots,f(a_n)}(t_m)) \in R_i \\
&\iff \mathfrak{U} \models_{f(a_1),\ldots,f(a_n)} \mathbf{R}_i(t_1, \ldots, t_m) \iff (f(a_1), \ldots, f(a_n)) \in R^\phi.
\end{aligned}$$

[12]We take as natural order between variables the one induced by their indexes.

- Suppose that $\phi = (\varphi \wedge \psi)$. Then:

$$\begin{aligned}
&(a_1, \ldots, a_n) \in R^\phi \iff \mathfrak{U} \models_{a_1,\ldots,a_n} (\varphi \wedge \psi)\\
&\iff \mathfrak{U} \models_{a_1,\ldots,a_n} \varphi \text{ and } \mathfrak{U} \models_{a_1,\ldots,a_n} \psi\\
&\iff (a_1, \ldots, a_n) \in R^\varphi \text{ and } (a_1, \ldots, a_n) \in R^\psi\\
&\iff (f(a_1), \ldots, f(a_n)) \in R^\varphi \text{ and } (f(a_1), \ldots, f(a_n)) \in R^\psi\\
&\iff \mathfrak{U} \models_{f(a_1),\ldots,f(a_n)} \varphi \text{ and } \mathfrak{U} \models_{f(a_1),\ldots,f(a_n)} \psi\\
&\iff \mathfrak{U} \models_{f(a_1),\ldots,f(a_n)} (\varphi \wedge \psi) \iff (f(a_1), \ldots, f(a_n)) \in R^\phi.
\end{aligned}$$

- Suppose that $\phi = \neg\psi$. Then:

$$\begin{aligned}
&(a_1, \ldots, a_n) \in R^\phi \iff \mathfrak{U} \models_{a_1,\ldots,a_n} \neg\psi\\
&\iff \text{not } \mathfrak{U} \models_{a_1,\ldots,a_n} \psi \iff (a_1, \ldots, a_n) \notin R^\psi \iff (f(a_1), \ldots, f(a_n)) \notin R^\psi\\
&\iff \text{not } \mathfrak{U} \models)_{f(a_1),\ldots,f(a_n)}\psi \iff \mathfrak{U} \models_{f(a_1),\ldots,f(a_n)} \neg\psi\\
&\iff (f(a_1), \ldots, f(a_n)) \in R^\phi.
\end{aligned}$$

- Finally, suppose that $\phi = \forall \mathbf{x}\psi$. Then:

$$\begin{aligned}
&(a_1, \ldots, a_n) \in R^\phi \iff \mathfrak{U} \models_{a_1,\ldots,a_n} \forall \mathbf{x}\psi\\
&\iff \text{for all } u \in U,\ \mathfrak{U} \models_{(a_1,\ldots,a_n)[\mathbf{x}/u]} \psi \iff (a_1, \ldots, a_n)[\mathbf{x}/u] \in R^\psi;
\end{aligned}$$

  now, $(a_1, \ldots, a_n)[\mathbf{x}/u]$ is a vector which coincides with $(a_1, \ldots, a_n)$, except (possibly) for the position corresponding to the free variable $\mathbf{x}$ of ϕ, where it has value u. But then:

$$\begin{aligned}
&(a_1, \ldots, a_n)[\mathbf{x}/u] \in R^\psi \iff (f(a_1), \ldots, f(a_n))[\mathbf{x}/f(u)] \in R^\psi\\
&\iff \text{for all } u \in U,\ \mathfrak{U} \models_{(f(a_1),\ldots,f(a_n))[\mathbf{x}/u]} \psi\\
&\iff (\text{since } f \text{ is bijective}) \text{ for all } u \in U,\ \mathfrak{U} \models_{(f(a_1),\ldots,f(a_n))[\mathbf{x}/u]} \psi\\
&\iff \mathfrak{U} \models_{f(a_1),\ldots,f(a_n)} \forall \mathbf{x}\psi \iff (f(a_1), \ldots, f(a_n)) \in R^\phi.
\end{aligned}$$

This completes the proof. □

Relativistic simultaneity is not an equivalence relation

With the help of Theorem 2.2.1, we shall now be able to prove that no non-trivial equivalence relation is first-order definable in the space-time starting only from precedence relation. This explains why the previously defined simultaneity (which is an intransitive relation) is the best possible definition of concurrency in a relativistic setting. Our results are actually a rephrasing (and a bland generalization) of Theorem I.2.1.5 of [Ben91].

Theorem 2.2.2 *No non-trivial*[13] *equivalence relation is first-order definable on the structure* $(\mathbb{Q}, <)$ *(where* $<$ *is the standard linear order for the rationals).*

Before proving Theorem 2.2.2, we need the following

Lemma 2.2.1 *Let* $q, q', q'' \in \mathbb{Q}$ *be three distinct rationals, with* $q' < q \iff q'' < q$*. There exists an automorphism* $f_q^{q' \to q''}$ *of* $(\mathbb{Q}, <)$ *such that* $f_q^{q' \to q''}(q) = q$ *and* $f_q^{q' \to q''}(q') = q''$*.*

[13] An equivalence relation is trivial if it is either the identity or the universal relation.

Proof: Let $f = f_q^{q' \to q''}$ be defined as follows:

$$f(x) = \frac{(q'' - q)x + (q' - q'')q}{q' - q}.$$

It is immediate to prove that, under the condition assumed on q, q', q'', we have $x < y$ if and only if $f(x) < f(y)$. A straightforward check proves that in fact $f(q) = q$ and $f(q') = q''$. $\square$

Now we can come to the proof of the theorem:

Proof of Theorem 2.2.2: Let $\sim$ be a definable equivalence relation different from the identity, and suppose that $x \sim y$ and $x \neq y$ (say, $x < y$). By Theorem 2.2.1, $\sim$ is invariant under every automorphism of $(\mathbb{Q}, <)$ and so, in particular, under the automorphisms defined in Lemma 2.2.1. Take any $z \neq y$ with $x < z$: since $x \sim y$ we have $f_x^{y \to z}(x) \sim f_x^{y \to z}(y)$, which means $x \sim z$. Moreover, for any $z < x$ we have $f_y^{x \to z}(x) \sim f_y^{x \to z}(y)$, i.e., $z \sim y$. Using transitivity, we thus have that every point is related to x under $\sim$, i.e., $\sim$ is the universal relation. $\square$

By using exactly the same arguments, one can show that the same happens for $(\mathbb{R}, <)$. We now pass to the Minkowski two-dimensional space, taking $\mathbb{Q}$ as underlying field (but the same is true for $\mathbb{R}$); this is exactly Theorem I.2.1.5 of [Ben91], even though our proof is slightly different, and uses Theorem 2.2.2.

Theorem 2.2.3 (Van Benthem [Ben91]) *No non-trivial equivalence relation is first-order definable on the structure $(\mathbb{Q} \times \mathbb{Q}, <)$, with $<$ defined componentwise.*

Proof: First observe that, if f, g are automorphisms of $(\mathbb{Q}, <)$, then

$$\begin{array}{rccc} \langle f, g \rangle : & \mathbb{Q} \times \mathbb{Q} & \to & \mathbb{Q} \times \mathbb{Q} \\ & (x, y) & \mapsto & (f(x), g(y)) \end{array}$$

is an automorphism of $(\mathbb{Q} \times \mathbb{Q}, <)$; in fact $(x, y) < (x', y')$ iff $x < x'$ and $y < y'$, which happens iff $f(x) < f(x')$ and $g(x) < g(x')$, i.e., $\langle f, g \rangle(x, y) < \langle f, g \rangle(x', y')$. Now, by combining in the various possible ways the identity map $\mathbf{1}_{\mathbb{Q}}$ and the automorphisms of Lemma 2.2.1, and by using the same arguments as in the proof of Theorem 2.2.2, we obtain the result. $\square$

In conclusion, the way we used to define simultaneity is not only satisfactory and intuitive, but there is no way of defining it as an equivalence relation by starting only from relativistic causality (precedence). In other words, simultaneity is unavoidably a proper tolerance relation (and not an equivalence), at least if we allow only for quantification over space-time points in our metalanguage, as the above quotation suggests.

Chapter 3

An introduction to partial orders and domains

In this chapter, we shall introduce some basic notions of domain theory. Some of the results presented here are quite standard, and can be found for example in the survey [GS90]; some other theorems are new, and will be useful in the sequel. We shall prove all the results which are non-trivial or which cannot be found directly in the literature.

3.1 Fundamentals

A *preorder* (or quasi-order) on a set P is a relation $\sqsubseteq$ which is reflexive and transitive. The pair $\mathcal{P} = (P, \sqsubseteq)$ is called a preordered set; when no confusion arises, we use P (D) and $\sqsubseteq$ (possibly, with special subscripts) to denote the underlying set and preorder relation of the preordered set $\mathcal{P}$ ($\mathcal{D}$, respectively). If $\sqsubseteq$ is also antisymmetric, i.e. if

$$\forall x, y \in P. \, (x \sqsubseteq y \ \wedge \ y \sqsubseteq x) \implies x = y$$

then we say that $\sqsubseteq$ is a *(partial) order* on P, and that $\mathcal{P}$ is a partially ordered set (or *poset*, for short). If moreover for all $x, y \in P$ either $x \sqsubseteq y$ or $y \sqsubseteq x$ holds, we say that $\sqsubseteq$ is a *total (or linear) order*.

The *covering relation* associated with $\sqsubseteq$, usually denoted by $\sqsubset\!\cdot$, is defined by putting $x \sqsubset\!\cdot y$ iff

$$x \neq y \ \wedge \ x \sqsubseteq y \ \wedge \ \forall z. \, (x \sqsubseteq z \sqsubseteq y \implies x = z \ \vee \ z = y).$$

In other words, $\sqsubset\!\cdot = \sqsubseteq \setminus \sqsubseteq^2$; we say that $\sqsubseteq$ is *combinatorial* if it coincides with the reflexive and transitive closure of[1] $\sqsubset\!\cdot$.

Now, let $\mathcal{P}$ be a poset, $X \subseteq P$ and $p \in P$; we shall say that

- p is an *upper bound* (lower bound) for X, and we write $X \sqsubseteq p$ ($p \sqsubseteq X$, respectively), iff $x \sqsubseteq p$ ($p \sqsubseteq x$, resp.) for all $x \in X$; if X has an upper (lower) bound, we say that it is *compatible* (lower compatible, resp.), and write it as $X \uparrow$ ($X \downarrow$, respectively);

[1] In general, it is properly included in it. For example, the poset $\mathbb{Q}$ of rationals w.r.t. their natural ordering gives rise to an empty covering relation.

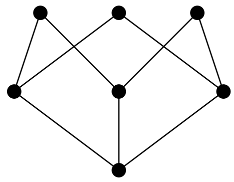


Figure 3.1: A consistently complete poset which is not coherent

- p is the *least upper bound* (greatest lower bound) for X, and we write $p = \sqcup X$ ($p = \sqcap X$, resp.), iff p is an upper (lower) bound for X and, if $p' \in P$ is another upper (lower) bound for X, then $p \sqsubseteq p'$ ($p' \sqsubseteq p$, resp.); if moreover $p \in X$, we say that p is the *maximum* (minimum, resp.) of X, and sometimes write $\max X$ ($\min X$) instead of $\sqcup X$ ($\sqcap X$);
- X is *directed* iff it is not empty and for any two $x, x' \in X$ there is some $x'' \in X$ such that $\{x, x'\} \sqsubseteq x''$.

It is customary to use the notations $x \uparrow y$, $x_1 \sqcup x_2 \sqcup \ldots \sqcup x_n$ and $x_1 \sqcap x_2 \sqcap \ldots \sqcap x_n$ as abbreviations for $\{x, y\} \uparrow$, $\sqcup\{x_1, \ldots, x_n\}$ and $\sqcap\{x_1, \ldots, x_n\}$, respectively. Moreover, for every $A \subseteq P$ we let $\downarrow A$ ($\uparrow A$) denote the set of lower bounds (upper bounds, respectively) of A (in the special case when A is a singleton, brackets are omitted).

A *precpo* (pre-complete partial order) is a poset $\mathcal{P}$ such that, for every $D \subseteq P$ which is directed, $\sqcup D$ exists. In particular, a *cpo* (complete partial order) is a precpo which contains a minimum element, usually indicated by $\perp$, and called *bottom*.

If every compatible subset of P has a least upper bound, we say that the poset is *consistently complete* (other terminology: Dedekind-complete). A stronger condition is coherence; a subset $X \subseteq P$ is *pairwise compatible* if and only if for any two elements $x, y \in X$ it happens that $x \uparrow y$. If every pairwise compatible subset of P has a least upper bound, we say that the poset is *coherent*. Note that clearly every coherent poset is consistently complete (because a compatible set is also pairwise compatible), but the converse is not true, as witnessed by the poset represented[2] in Fig. 3.1.

It is rather easy to prove that, if $\mathcal{P}$ is a consistently complete poset, then every non-empty subset has a greatest lower bound: this happens because the greatest lower bound of a set is simply the least upper bound of the set of lower bounds.

If $\mathcal{P}$ and $\mathcal{P}'$ are two cpo's, we say that a function $f : P \to P'$ is

- *monotone* if $x \sqsubseteq y$ implies $f(x) \sqsubseteq f(y)$ (i.e., f is a poset homomorphism);
- *continuous* if it is monotone, and moreover, for every directed set $D \subseteq P$, it holds that $f(\sqcup D) = \sqcup f(D)$ (the RHS is well-defined, because $f(D) = \{f(d), d \in D\}$ is clearly a directed subset of P', if f is monotone);

[2] A poset is usually represented using its Hasse diagram; in a Hasse diagram, the elements are drawn as points, and $x \sqsubseteq y$ holds iff there is an ascending path going from the point representing x to the point representing y. To be more precise, the Hasse diagram is a conventional way to represent the covering relation associated to the partial order.

- *strict* if $f(\bot) = \bot'$;
- an *isomorphism* if it is monotone and there exists a monotone function $g : P' \to P$ such that $g \circ f = \mathbf{1}_P$ and $f \circ g = \mathbf{1}_{P'}$; or, equivalently, if f is one-to-one, onto and for any two elements $x, y \in P$ it holds that $x \sqsubseteq y \iff f(x) \sqsubseteq' f(y)$;
- *additive* if, whenever $x \sqcup y$ exists, also $f(x) \sqcup f(y)$ exists, and $f(x \sqcup y) = f(x) \sqcup f(y)$.

Note that one can define an order between functions: given two functions $f, g : P \to P'$, define $f \sqsubseteq g$ if and only if $f(x) \sqsubseteq g(x)$ holds for all $x \in P$. This is called the *pointwise* ordering of functions. We usually denote the poset of continuous (strict and continuous) functions between two posets $\mathcal{P}, \mathcal{P}'$ (with the pointwise ordering defined before) by $[\mathcal{P} \to \mathcal{P}']$ (respectively: $[\mathcal{P} \to_\bot \mathcal{P}']$).

Let $\mathcal{P}$ be a cpo; an element $x \in P$ is *compact* (or isolated) iff for every directed set $D \subseteq P$ it holds that

$$x \sqsubseteq \sqcup D \implies \exists d \in D.\, x \sqsubseteq d.$$

The usual way one should think of a compact element is by interpreting it as a "finite" approximation. In fact, we can just imagine that the order describes the information content of each element: an isolated element x is one which has only a finite information content, because every time we can gather all the information of x, we can do it also in a finite fashion.

The set of compact elements of P is denoted by P°; we define, for all the elements x of P, the set $K(x) = \downarrow x \cap P^\circ$. The cpo $\mathcal{P}$ is *algebraic* iff for every $x \in P$, the set $K(x)$ is directed, and moreover $\sqcup K(x) = x$. It is *ω-algebraic* if moreover P° is countable.

A *domain* is an algebraic cpo; it is a *Scott domain* if it is moreover consistently complete. Here are some properties concerning the relations between compact elements and continuity of functions between domains:

Property 3.1.1 *Let $\mathcal{D}_0, \mathcal{D}_1$ be two domains:*

1. *a continuous function is uniquely identified by its restriction on the set of compact elements; in other words, if $f, g : \mathcal{D}_0 \to \mathcal{D}_1$ are continuous and $f(d) = g(d)$ for all $d \in D_0^\circ$, then $f = g$;*
2. *a function $f : \mathcal{D}_0 \to \mathcal{D}_1$ is continuous[3] iff for all $x \in D_0$ and all $b \in D_1^\circ$*

$$b \sqsubseteq_1 f(x) \iff \exists a \in K(x).\, b \sqsubseteq_1 f(a);$$

3. *for every monotone function $f : D_0^\circ \to D_1^\circ$ there is exactly one continuous function $h : \mathcal{D}_0 \to \mathcal{D}_1$ extending f*

[3]This is also known as the $\epsilon - \delta$ version of continuity; in fact, it can be informally stated as follows: if b is a finite approximation of $f(x)$ then there is a finite approximation a of x, whose image has also b as a finite approximation. In other words, it is always possible to find a sufficiently "good" approximated input to a continuous function, if we just need an approximated result.

Proof: (1) Let $x \in D_0$; then $x = \sqcup K(x)$, and so $f(x) = f(\sqcup K(x))$. By continuity of f we obtain $f(x) = \sqcup f(K(x))$. But now, f and g coincide on the set of compact elements; thus $f(K(x)) = g(K(x))$ and so $f(x) = \sqcup g(K(x)) = g(\sqcup K(x)) = g(x)$ as required.
(2) First suppose that f is continuous; the only non-trivial implication is $\Longrightarrow$ (the other follows by monotonicity). Now $b \sqsubseteq_1 f(\sqcup K(x)) = \sqcup f(K(x))$. But b is compact and $f(K(x))$ is a directed subset of D_1; so there must be an element $f(a)$ (with $a \sqsubseteq_0 x$ and a compact) such that $b \sqsubseteq_1 f(a)$, as required.
For the converse, we firstly prove that f is monotone. Suppose that $x \sqsubseteq_0 y$: if b is isolated in D_1 and such that $b \sqsubseteq_1 f(x)$, then there is some $a \in K(x)$ such that $b \sqsubseteq_1 f(a)$. But $a \sqsubseteq_0 y$ and $b \sqsubseteq_1 f(a)$, and so $b \sqsubseteq_1 f(y)$ (using the right-to-left implication of the hypothesis). Thus, for every compact element b one has $b \sqsubseteq_1 f(x) \Longrightarrow b \sqsubseteq_1 f(y)$, i.e., $K(f(x)) \subseteq K(f(y))$ and so $f(x) \sqsubseteq_1 f(y)$. Then, using algebraicity, we obtain that f must be monotone. Continuity is then proved analogously.
(3) Define $h(x) = \sqcup f(K(x))$, which is well-defined, since $\mathcal{D}_0$ is algebraic and f is monotone; clearly, h extends f (because, if d is compact, then $f(d) \in f(K(d))$ and so $h(d) = f(d)$). To prove that h is continuous, we shall use $\epsilon - \delta$ continuity. Let $x \in D_0$ and $b \in D_1^\circ$; since h is monotone, only the left-to-right implication needs to be proved. If $b \sqsubseteq_1 h(x) = \sqcup f(K(x))$ then, since b is compact, there must exist some compact element $a \in K(x)$ such that $b \sqsubseteq_1 f(a)$, as required. □

A very useful notion in domain theory is the concept of ideal, which allows one to complete a poset in order to obtain a domain. Let $\mathcal{P}$ be a poset with minimum element $\perp$; a *(directed) ideal* of $\mathcal{P}$ is a set $I \subseteq P$ such that

1. I is directed;
2. I is downward-closed, i.e., $\downarrow I = I$ (or, equivalently, if $x \in I$ and $y \sqsubseteq x$ then also $y \in I$).

In particular, for each $x \in P$, the set $\downarrow x$ is an ideal of $\mathcal{P}$, called the *principal ideal* generated by x. The set of ideals of $\mathcal{P}$ will be denoted by $\mathrm{Idl}(P)$, and the same notation will sometimes refer to the corresponding $\subseteq$-poset, which is usually called the "ideal completion" of $\mathcal{P}$.

The following result explains why ideals play an important rôle in the theory of domains:

Theorem 3.1.1 *Given a poset $\mathcal{P}$ with minimum, the poset Idl(P) is a domain whose compact elements are precisely the principal ideals of $\mathcal{P}$. Conversely, if $\mathcal{D}$ is a domain, the poset Idl(D°) is isomorphic to $\mathcal{D}$.*

Proof: For the first part, clearly $\mathrm{Idl}(P)$ has minimum $\{\perp\}$. If $S \subseteq \mathrm{Idl}(P)$ is a directed set, we prove that $I = \cup S$ is an ideal (and thus it is the least upper bound of S). Suppose $x \in I$ and $y \sqsubseteq x$; then $x \in J$ for some $J \in S$, and so $y \in J \subseteq I$. Now, suppose that $x, y \in I$; then $x \in J, y \in J'$ where $J, J' \in S$. But S is directed, and so there is some $J'' \in S$ such that $x, y \in J''$. But J'' is directed, and so there is some $z \in J'' \subseteq I$ such that $\{x, y\} \sqsubseteq z$, as required.
For the compact elements, if $\downarrow x \subseteq \cup S$, where S is directed, then in particular $x \in \cup S$ and so $\downarrow x$ will be included in some element of S. Conversely, suppose that I is a compact

ideal, and let S be the set of the principal ideals generated by the elements of I. This set is directed (in fact, if $\downarrow x, \downarrow y \in S$, then $x, y \in I$, and so there is some $z \in I$ such that $\downarrow x \subseteq \downarrow z$ and $\downarrow y \subseteq \downarrow z$), and clearly $\cup S = I$; so there is some element of S in which I is included, and thus I is a principal ideal. The fact that $\mathrm{Idl}(P)$ is algebraic follows directly from this observation.
For the second part, consider the usual function K which maps each element of D into the set of the compact elements below it. This is injective, because, by algebraicity, $K(x) = K(y)$ implies $\sqcup K(x) = \sqcup K(y)$ and hence $x = y$. It is also surjective: if I is an ideal, then let $x = \sqcup I$ (which exists, because I is a directed set). Clearly $K(x) = I$. So K is an order-isomorphism. □

This theorem in particular implies that every domain is uniquely determined by the poset of its compact elements, from which it can be completely recovered by ideal completion.

3.2 Stable functions, Berry's order and dI-domains

Let us go back for a while to the $\epsilon - \delta$ version of continuity (Property 3.1.1); recall that a function $f : \mathcal{D}_0 \to \mathcal{D}_1$ is continuous if and only if for all $x \in D_0$ and all $b \in D_1^\circ$

$$b \sqsubseteq_1 f(x) \iff \exists a \in K(x).\, b \sqsubseteq_1 f(a).$$

In other words, given any input x and any finite approximation b of the output determined by x, there exists a finite approximation of the input which leads to an approximation of the output "not worst than" b.

The problem here is that there is *no canonical* way to choose a: one would like to be able to choose the least possible such input, but in general there is no guarantee that such a minimum input exists. In other words, we would like to have a minimum finite element $M(f, x, b)$ which is not greater than x and whose image is not smaller than b. We shall now formalize this notion, and prove that it can be equivalently stated in a very simple way as a preservation of compatible greatest lower bounds.

Consider a continuous function $f : \mathcal{D}_0 \to \mathcal{D}_1$; we say that f satisfies the *minimum modulus property* iff for all $x \in D_0$ and $y \in K(f(x))$, the set[4]

$$A(f, x, y) = \{e \in K(x) : y \sqsubseteq f(e)\}$$

has a least element, which is denoted by $M(f, x, y)$.

We shall be interested in considering stable functions between domains of a very special kind, the so-called "dI-domains". A domain $\mathcal{D}$ is *finitary* iff $K(x)$ is finite for every $x \in D^\circ$; an example of non-finitary domain is represented by the ordinal $\omega + 2$ (see Fig. 3.2): this is clearly a domain, whose compact elements are all finite ordinals plus the element $\omega + 1$, which has infinitely many compact elements below.

A domain $\mathcal{D}$ is *distributive* iff, for any three elements $x, y, z \in D$, if $y \uparrow z$ then $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$. A finitary distributive Scott domain is often called *dI-domain*, a term introduced by Berry in [Ber79].

[4] Note that this set is non-empty, because of Property 3.1.1 ($\epsilon - \delta$ continuity).

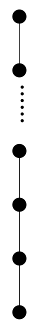


Figure 3.2: The ordinal $\omega + 2$

In the case of dI-domains, we can give a definition which is equivalent to the minimum modulus property:

Lemma 3.2.1 *Let $f : \mathcal{D}_0 \to \mathcal{D}_1$ be a continuous function between dI-domains. Then, the following are equivalent:*

1. *f satisfies the minimum modulus property;*
2. *for all $x, x' \in D_0$, if $x \uparrow x'$ then $f(x \sqcap x') = f(x) \sqcap f(x')$.*

Proof: The proof is simple but lengthy. We refer the interested reader to [BG92], Proposition 3.10. □

A function which is continuous and satisfies the condition(s) of Lemma 3.2.1 is called *stable*. The notion of stability was introduced by Berry [Ber78], with the aim of generalizing the notion of sequentiality at higher types.

A natural question is whether we can use the minimum modulus property to define a new notion of "order" among stable functions, which is finer than the usual pointwise order. Suppose that $f, g : \mathcal{D}_0 \to \mathcal{D}_1$ are two stable functions, with $f \sqsubseteq g$ (under the usual pointwise ordering): intuitively, this is taken to mean that f is an approximation of g, i.e., it always gives an output which is an approximation to that given by g under the same input.

Let now $x \in D_0$ and $y \in K(f(x))$: there will be a least approximation $M(f, x, y)$ of the input x which furnishes an output which is approximated by y. Clearly $M(g, x, y) \sqsubseteq M(f, x, y)$, because g always gives better outputs, but it could be the case that a poorer approximation is enough for g to get the same result. If this does not happen, i.e., if

$$f \sqsubseteq g \text{ and } \forall x \in D_0, y \in K(f(x)).\, M(f, x, y) = M(g, x, y)$$

then we say that f is stably less than g, and write $f \sqsubseteq_s g$. As a matter of fact, $\sqsubseteq_s$ can be characterized for dI-domains in the following way:

Lemma 3.2.2 *For any two stable functions $f, g : \mathcal{D}_0 \to \mathcal{D}_1$ between dI-domains, with $f \sqsubseteq g$,*

$$f \sqsubseteq_s g \iff (\forall x', x \in D_0.\, x' \sqsubseteq x \implies f(x') = f(x) \sqcap g(x')).$$

Proof: For a proof, see [BG92], Proposition 4.7. □

The relation $\sqsubseteq_s$ is actually a partial order (called the *Berry order*) which refines the pointwise one, as stated in the following

Property 3.2.1 *$\sqsubseteq_s$ is a partial order relation, and $f \sqsubseteq_s g$ implies $f \sqsubseteq g$.*

Proof: First note that, whenever $f \sqsubseteq_s g$, for all $a \in D_0$, $f(a) = f(a) \sqcap g(a) \sqsubseteq g(a)$. So $f \sqsubseteq_s g$ implies $f \sqsubseteq g$. Reflexivity is obvious: if $a' \sqsubseteq a$ then $f(a') \sqsubseteq f(a)$, and so $f(a') = f(a) \sqcap f(a')$. For the antisymmetric property, suppose $f \sqsubseteq_s g$ and $g \sqsubseteq_s f$, and take any $a \in D_0$. We have $f(a) = f(a) \sqcap g(a)$ and $g(a) = g(a) \sqcap f(a)$, thus $f(a) = g(a)$. For transitivity, suppose that $f \sqsubseteq_s g$ and $g \sqsubseteq_s h$, and let $a' \sqsubseteq a$. We then have $f(a') = f(a) \sqcap g(a') = f(a) \sqcap g(a) \sqcap h(a')$ which is in turn equal to $f(a) \sqcap h(a')$, because $f \sqsubseteq_s g \implies f \sqsubseteq g$. □

The set of stable functions from $\mathcal{D}_0$ to $\mathcal{D}_1$, ordered by the Berry order, is denoted by $[\mathcal{D}_0 \to_s \mathcal{D}_1]$ in the following.

3.3 Stable embedding-projection pairs

It is possible to define a notion of embedding that can be interpreted as an approximation relation between domains, roughly in the same sense as the order relation of a domain is taken to define an approximation relation between (the information content of) its elements.

The standard way to do this consists in defining the so-called embedding-projection pairs. An *embedding-projection pair* (or EPP for short) between two domains $\mathcal{D}_0, \mathcal{D}_1$ is a pair of functions $\langle f, g\rangle : \mathcal{D}_0 \to \mathcal{D}_1$, where $f : \mathcal{D}_0 \to \mathcal{D}_1$ (the "embedding") and $g : \mathcal{D}_1 \to \mathcal{D}_0$ (the "projection") are both continuous, and satisfy $g \circ f = \mathbf{1}_{\mathcal{D}_0}$ and $f \circ g \sqsubseteq \mathbf{1}_{\mathcal{D}_1}$. Notice that:

Property 3.3.1 *Let $\langle f, g\rangle : \mathcal{D}_0 \to \mathcal{D}_1$ be an embedding-projection pair. Then f is injective, g is surjective and both are strict.*

Proof: If $f(x) = f(y)$ then also $g(f(x)) = g(f(y))$ and (since $g \circ f = \mathbf{1}$) $x = y$. Let now $x \in D_0$; then $g(f(x)) = x$ and so $x \in g(D_1)$. Thus, g is surjective. For strictness of f, just observe that $\perp_1 \sqsubseteq_1 f(\perp_0)$, which implies $f(g(\perp_1)) = \perp_1$, and so, since $\perp_0 \sqsubseteq_0 g(\perp_1)$, we have $f(\perp_0) = \perp_1$ also, so f is strict. Moreover, $g(\perp_1) = g(f(\perp_0)) = \perp_0$, and thus g is also strict. □

We will mainly be interested in a specialized version of this notion, introduced by Kahn and Plotkin in [KP78] (see also Curien [Cur86]). An embedding-projection pair $\langle f, g\rangle : \mathcal{D}_0 \to \mathcal{D}_1$ is a *stable embedding-projection pair* (or SEPP for short) if moreover, for all $x_0 \in D_0$ and $x_1 \in D_1$, if $x_1 \sqsubseteq_1 f(x_0)$ then $x_1 = f(g(x_1))$.

There is a strict relation between SEPP's, stability and Berry's order, which is made clear in the following

Lemma 3.3.1 *Let $\langle e, p\rangle : \mathcal{D}_0 \to \mathcal{D}_1$ be an embedding-projection pair. The following are equivalent:*

1. *$\langle e, p\rangle$ is a stable embedding-projection pair;*

2. *e and p are stable, and $e \circ p \sqsubseteq_s \mathbf{1}$.*

Proof: (1) $\implies$ (2) We first prove that p is stable; suppose that $y, y' \in D_1$ with $y \uparrow y'$. By monotonicity, $p(y \sqcap y') \sqsubseteq p(y) \sqcap p(y')$, so we need only prove the converse. By monotonicity, once again, we obtain $e(p(y) \sqcap p(y')) \sqsubseteq e(p(y)) \sqsubseteq y$, and likewise $e(p(y) \sqcap p(y')) \sqsubseteq y'$, so

$$e(p(y) \sqcap p(y')) \sqsubseteq y \sqcap y'.$$

Now, applying p to both sides, and using $p \circ e = \mathbf{1}$, we get $p(y) \sqcap p(y') \sqsubseteq p(y \sqcap y')$.
For proving stability of e, let $x, x' \in D_0$ with $x \uparrow x'$. We have to prove that $e(x) \sqcap e(x') = e(x \sqcap x')$. Note that $e(x) \sqsubseteq e(x \sqcup x')$ as well as $e(x') \sqsubseteq e(x \sqcup x')$; so we get $e(x) \sqcap e(x') \sqsubseteq e(x \sqcup x')$, and thus, using the definition of stable embedding-projection pair, $e(x) \sqcap e(x') = e(p(e(x) \sqcap e(x')))$ which, by stability of p, equals $e(p(e(x)) \sqcap p(e(x'))) = e(x \sqcap x')$.
We should now prove that $e \circ p \sqsubseteq_s \mathbf{1}$. Suppose that $y, y' \in D_1$ with $y \uparrow y'$. Clearly $e(p(y')) \sqsubseteq e(p(y)) \sqcap y'$, so we need only prove the converse. Suppose that $y'' \sqsubseteq e(p(y)) \sqcap y'$: by definition of stable embedding-projection pair, $y'' \sqsubseteq e(p(y))$ implies $y'' = e(p(y''))$ and thus, by monotonicity of $e \circ p$, we have $y'' \sqsubseteq e(p(y'))$. Thus $y'' \sqsubseteq e(p(y)) \sqcap y'$ implies $y'' \sqsubseteq e(p(y'))$, and we are done.
(2) $\implies$ (1) Suppose that $y \sqsubseteq e(x)$. Then, since $e \circ p \sqsubseteq_s \mathbf{1}$, one has $(e \circ p)(y) = (e \circ p)(e(x)) \sqcap y$, i.e., $e(p(y)) = e(p(e(x))) \sqcap y = e(x) \sqcap y = y$. $\square$

Thus, we can simply say that SEPP's are just common embedding-projections provided that we consider only stable (instead of continuous) functions, and Berry's order (instead of the pointwise one).

One can define the composition of two (stable) embedding-projection pairs $\langle f, g\rangle : \mathcal{D}_0 \to \mathcal{D}_1$ and $\langle f', g'\rangle : \mathcal{D}_1 \to \mathcal{D}_2$ as the pair $\langle f' \circ f, g \circ g'\rangle$: a routine check shows that this is in fact a (S)EPP.

There is a close relation between stable embedding-projection pairs and special subsets of Scott domains called strong ideals. If $\mathcal{D}$ is a Scott domain, a *strong ideal* $X \subseteq D$ is a set satisfying the following:

1. X is not empty, and downward-closed;

2. if $x, y \in X$ and $x \uparrow y$, then $x \sqcup y \in X$; i.e., X is closed under taking least upper bounds of finite (compatible) sets;

3. if $S \subseteq X$ is directed, then $\sqcup S \in X$; i.e., X is closed under taking least upper bounds of directed sets.

An important property of strong ideals is the following (for an alternative proof, see Kahn and Plotkin [KP78], Proposition P-8.6):

Property 3.3.2 *Let X be a strong ideal of the Scott domain $\mathcal{D}$. Then X, with the inherited ordering, is also a Scott domain, which is coherent if $\mathcal{D}$ is, whose compact elements are precisely those compact elements of $\mathcal{D}$ which are contained in X.*

Proof: X contains $\bot$, which is its minimum element (this follows since $X \neq \emptyset$ and it is downward-closed). If $S \subseteq X$ is directed, then $\sqcup S \in X$, and so X is a cpo.
Now, suppose that $x \in X \cap D^\circ$: we prove that x is also compact in X. If $S \subseteq X$ is directed and $x \sqsubseteq \sqcup_X S = \sqcup_D S$, then $x \sqsubseteq s$ for some $s \in S$, because $x \in D^\circ$. On the contrary, suppose that x is compact in X: we prove that it is also compact in D. Let $S \subseteq D$ be directed with $x \sqsubseteq \sqcup S$, and let $T = \{a \in X : \exists s \in S.\, a \sqsubseteq s\}$. T is directed: if $a, b \in T$ then there exist $s, s' \in S$ with $a \sqsubseteq s$ and $b \sqsubseteq s'$; by directedness of S, there is some $s'' \in S$ such that $\{a, b\} \sqsubseteq s''$, so $a \uparrow b$ and thus $a \sqcup b \in T$. Now $x \sqsubseteq \sqcup S$ implies that $x \sqsubseteq s$ holds for all $s \in S$. So $x \in T$ and thus $x \sqsubseteq \sqcup T$. But x is compact in X, and therefore $x \sqsubseteq a$ for some $a \in T$. Thus also $x \in D^\circ$.
Now, algebraicity is easily proved. For all $x \in X$, the set $K_X(x) = K(x) \cap X$ is directed; so $\sqcup K_X(x) = \sqcup(K(x) \cap X) = x$, as required. For consistent completeness, let $A \sqsubseteq X$ be a compatible set. Take $B = \downarrow(\sqcup A) \cap X$: this set is directed and so $\sqcup B \in X$; but then $\sqcup B = \sqcup A$. (For coherence, the proof is analogous). $\square$

The aforementioned relation between strong ideals and SEPP's is expressed by the following

Proposition 3.3.1 ([BCS93]; see also [KP78]) *If $\langle f, g\rangle : \mathcal{D}_0 \to \mathcal{D}_1$ is a stable embedding-projection pair between Scott domains, then $f(D_0)$ is a strong ideal of $\mathcal{D}_1$. Conversely, if X is a strong ideal of $\mathcal{D}$, the inclusion map $i : X \hookrightarrow D$ and the function $p : D \to X$ defined by*

$$p(d) = \sqcup(K(d) \cap X)$$

form a stable embedding-projection pair $\langle i, p\rangle$ from X (with the ordering inherited from $\mathcal{D}$) to D.

Proof: First part. For downward closure, if $y \sqsubseteq_1 f(x)$ then by stability $y = f(g(y))$ and so $y \in f(D_0)$. Suppose now that $f(x) \uparrow f(y)$ in $\mathcal{D}_1$. Then let $z = f(g(f(x) \sqcup_1 f(y)))$; since $f \circ g \sqsubseteq \mathbf{1}$, we have $z \sqsubseteq_1 f(x) \sqcup_1 f(y)$. But $f(x) \sqsubseteq_1 f(x) \sqcup f(y)$ and so $g(f(x)) \sqsubseteq_0 g(f(x) \sqcup f(y)$ (because g is monotone) and finally $f(g(f(x))) \sqsubseteq_1 z$ (because f is monotone), which means $f(x) \sqsubseteq_1 z$; analogously, also $f(y) \sqsubseteq_1 z$ and so $f(x) \sqcup f(y) \sqsubseteq_1 z$. Thus finally $z = f(x) \sqcup f(y)$, and $z \in f(D_0)$. Finally, suppose that $S \subseteq f(D_0)$ is a directed set; in exactly the same way one proves that $\sqcup_1 S = f(g(\sqcup_1 S))$, and so $f(D_0)$ is a strong ideal.
Second part. The inclusion map is clearly continuous, and so is p. Now $p(i(x)) = p(x) = \sqcup(K(x) \cap X) = x$ and conversely $i(p(x)) = \sqcup(K(x) \cap X) \sqsubseteq \sqcup K(x) = x$. Finally, stability follows from downward-closure of X. $\square$

Since embeddings are injective, the first part of Proposition 3.3.1 simply states that, if $\langle f, g\rangle : \mathcal{D}_0 \to \mathcal{D}_1$ is a stable embedding-projection pair, then $f(D_0)$ is isomorphic (via f) to a strong ideal of $\mathcal{D}_1$: in fact, if $f(x) \sqsubseteq f(y)$ then $g(f(x)) \sqsubseteq g(f(y))$ and so $x \sqsubseteq y$.

In the case of coherent domains, SEPP's can be characterized in a very simple way:

Lemma 3.3.2 (See [BCS93]) *Let $\mathcal{D}_0, \mathcal{D}_1$ be coherent domains, and $f : \mathcal{D}_0 \to \mathcal{D}_1$ be a continuous additive injection satisfying the conditions:*

1. *if $f(x) \uparrow f(x')$ then $x \uparrow x'$;*

2. *for every $y \in D_1$ and every $x \in D_0$, if $y \sqsubseteq f(x)$ then $y = f(x')$ for some $x' \in D_0$.*

Then f is a stable embedding, i.e., there exists $g : \mathcal{D}_1 \to \mathcal{D}_0$ such that $\langle f, g\rangle$ is a stable embedding-projection pair.

Proof: It is straightforward to check that f is an (order) isomorphism between D_0 and $f(D_0) \subseteq D_1$. In fact, if $f(x) \sqsubseteq f(y)$ then $f(x) \uparrow f(y)$ and so, by the first condition, $x \uparrow y$. But then, by additivity, $f(x \sqcup y) = f(x) \sqcup f(y) = f(y)$ and thus, by injectivity, $x \sqcup y = y$, i.e., $x \sqsubseteq y$. Moreover, $f(D_0)$ is a strong ideal of $\mathcal{D}_1$, and so the result can be obtained by using Proposition 3.3.1. Indeed, the pair $\langle i, p\rangle : f(D_0) \to D_1$ is a SEPP, and moreover $\langle f, f^{-1}\rangle : D_0 \to f(D_0)$ is also a SEPP (in fact, an isomorphism). So, their composition $\langle i \circ f, f^{-1} \circ p\rangle$ is a SEPP (note that the embedding is really $i \circ f = f$). □

The following proposition explains the structure of ideal completions which are strong ideals in a Scott domain.

Proposition 3.3.2 (See [BCS93], Proposition 2.2.4) *Let $\mathcal{P}$ be a consistently complete poset with minimum, and X be a non-empty subset of P such that:*

1. *X is downward-closed;*
2. *if $x, y \in X$ and $x \uparrow y$, then $x \sqcup y \in X$.*

The domain $Idl(X)$ is a strong ideal of $Idl(P)$. □

3.4 Atomicity and dI-domains

There is a special property concerning strong ideals of dI-domains, which was stated in [BCS93], and is an adaptation of the proof of Proposition 2.3.7 in Curien [Cur86].

Property 3.4.1 *Let $\mathcal{D}$ be a dI-domain, and X a strong ideal of $\mathcal{D}$. Then X, with the inherited ordering, is also a dI-domain, whose compact elements are precisely those compact elements of $\mathcal{D}$ which are contained in X.*

Proof: We already know from Property 3.3.2 that X is a Scott domain and $X^\circ = D^\circ \cap X$; so the domain is also finitary. Distributivity is straightforward. □

In a Scott domain $\mathcal{D}$, an element d is a *complete prime* if and only if, for every compatible set $X \subseteq D$ it holds that

$$d \sqsubseteq \sqcup X \implies \exists x \in X.\, d \sqsubseteq x.$$

The set of complete primes is denoted by $\mathrm{Pr}(D)$. We say that $\mathcal{D}$ is *prime algebraic* iff $x = \sqcup K^P(x)$ holds for each $x \in D$, where

$$K^P(x) = \{d \in \mathrm{Pr}(D) : d \sqsubseteq x\}.$$

The following theorem gives a precise relation between prime algebraicity and distributivity in finitary Scott domains:

Theorem 3.4.1 (Winskel [Win87]) *Let $\mathcal{D}$ be a finitary Scott domain. Then $\mathcal{D}$ is distributive (i.e., a dI-domain) iff it is prime algebraic.* □

Given a cpo $\mathcal{D}$, an *atom* is any element $d \in D$ such that $\perp \sqsubset\!\cdot d$. The set of atoms is denoted by $D^\diamond$, and, for every $x \in D$, we let $K^A(x) =\downarrow x \cap D^\diamond$. A Scott domain $\mathcal{D}$ is *atomic* iff $\sqcup K^A(x) = x$ for every $x \in D$. Here is an interesting property of the atoms in a Scott domain:

Lemma 3.4.1 *Let $\mathcal{D}$ be a Scott domain.*

1. *Every atom of $\mathcal{D}$ is compact, i.e., $D^\diamond \subseteq D^\circ$;*
2. *if $\mathcal{D}$ is an atomic dI-domain, then for every compatible set of atoms A one has $K^A(\sqcup A) = A$.*

Proof: For the first part, if $x \in D^\diamond$ was not compact, since $x = \sqcup K(x)$, we would have $K(x) = \{\perp\}$ and thus $x = \perp$, contradicting the fact that x is an atom.
For the second part, the right-to-left inclusion is obvious, so we only have to prove the following statement: if a is an atom and $a \sqsubseteq \sqcup A$ then $a \in A$.
We first prove this in the case of finite A. Suppose by contradiction that $a \in A$: one has $a \sqcap (\sqcup A) = a$ (because $a \sqsubseteq \sqcup A$) but, by distributivity (A being finite)

$$a \sqcap (\sqcup A) = \sqcup\{a \sqcap x, x \in A\} = \perp$$

because $a \sqcap b = \perp$ for all $a, b \in D^\diamond, a \neq b$. So we have $a = \perp$, contradicting $a \in D^\diamond$.
For the infinite case, let $S = \{\sqcup B : B \subseteq A \text{ finite}\}$; this is clearly a directed set, and $\sqcup S = \sqcup A$. So $a \sqsubseteq \sqcup S$; but a is an atom, and so $a \sqsubseteq \sqcup B$ for some finite $B \subseteq A$. Using the finite-case part, we so have $a \in B \subseteq A$. □

Atomic dI-domains are also known as qualitative domains, since the work of Girard [Gir86] who first gave a nice and simple representation theory for them, which we shall sketch in some detail in Chapter 4.

An important property concerning the relation between atomicity and stable EPP's is the following:

Lemma 3.4.2 *Let $\langle f, g\rangle : \mathcal{D}_0 \to \mathcal{D}_1$ be a stable embedding-projection pair. If x is an atom of $\mathcal{D}_0$, then $f(x)$ is an atom of $\mathcal{D}_1$.*

Proof: First remember that f is injective and strict (by Property 3.3.1); so $f(x) \neq \perp_1$ (because x is an atom, and so it is different from $\perp_0$). Now, suppose $\perp_1 \sqsubseteq y \sqsubseteq f(x)$; since g is monotone, $g(\perp_1) = \perp_0 =\sqsubseteq g(y) \sqsubseteq g(f(x)) = x$. But then, since x is an atom, $\perp_0 = g(y)$ or $g(y) = x$. Since $y \sqsubseteq f(x)$, we have $y = f(g(y))$ and so either $f(\perp_0) = \perp_1 = y$ or $y = f(x)$. □

It is worth mentioning that atomic dI-domains have the special property of being locally isomorphic to complete boolean algebras, as stated in the following

Property 3.4.2 *Let $\mathcal{D}$ be an atomic dI-domain. For every $x \in D$, the set $\downarrow x$ (with the induced ordering) is a complete boolean algebra.*

Proof: The fact that $\downarrow x$ is a complete lattice is ensured by the consistent completeness of $\mathcal{D}$, and by the observation that $\downarrow x$ is upper bounded by x. Also, distributivity follows from the distributivity of $\mathcal{D}$. For complements, let $y \in\downarrow x$ and define $A = K^A(y)$ and $B = K^A(x) \setminus K^A(x)$. Observe that $\sqcup A = y$, and let y' be defined as $\sqcup B$. We have:

- $y \sqcup y' = (\sqcup K^A(y)) \sqcup (\sqcup(K^A(x) \setminus K^A(y))) = \sqcup K^A(x) = x$;
- $y \sqcap y' = (\sqcup K^A(y)) \sqcap (\sqcup(K^A(x) \setminus K^A(y)))$ which gives, by distributivity, $y \sqcap y' = \sqcup\{a \sqcap b : a \in K^A(y), b \in K^A(x) \setminus K^A(y)\}$. But clearly the greatest lower bounds of two different atoms is always the bottom, and so $y \sqcap y' = \perp$.

So y' is the complement of y. $\square$

Chapter 4

Some universal constructions

4.1 Introduction

In this chapter, we study the structure of tolerance spaces with the tools of universal algebra; to this end, we shall consider the category of (countable) tolerance spaces, with embeddings as morphisms. It is well-known, since Rado [Rad67], that this category (which is equivalent to the category of undirected graphs with rigid embeddings) contains a universal homogeneous object. We shall show that this category is equivalent to that of atomic coherent dI-domains (with stable embedding-projection pairs as objects, see Berry [Ber79]), and obtain as a consequence a universal homogeneous object for this category also.

Rado's very direct construction can be generalized, as to obtain universal homogeneous objects for other categories of representations. If one considers Girard's qualitative domains [Gir86], which are quite a natural generalization of tolerance spaces, it is rather easy to construct a universal homogeneous object, which is built much in the same way as Rado's graph. We shall prove that this category is equivalent to that of atomic dI-domain, and thus obtain a very direct construction of a universal homogeneous domain of this kind.

It is immediate to observe that these constructions can be further generalized considering structures with a notion of causality (enabling), like event structures [Win80, NPW81]: clearly, tolerance spaces (qualitative domains) are very simple cases of prime event structures (general event structures, respectively), where the enabling relation is trivial. We shall further generalize our constructions, and obtain a very direct definition for the universal homogeneous stable event structure. Unfortunately, in this case we have no categorical equivalence with the corresponding category of domains (i.e., the dI-domains). So, only a universal domain can be obtained in this way, but homogeneity cannot be insured.

We shall also provide an alternative construction for a universal homogeneous dI-domain, presented in [BCS93], based on the notion of Mazurkiewicz's trace, and still using Rado's graph as starting point.

The work contained in this chapter can be interpreted in two ways: on one side, we wish to show how tolerance spaces are a special case of very well-known representations; on the other, we prove that their universal structure is especially well-behaved.

4.2 An introduction to universality

In the theory of denotational semantics of programming languages and concurrency, many authors have established the (in)existence of particular kinds of universal domains. The pioneering work of Scott [Sco76], which provided a universal ω-algebraic lattice, has been followed by more research in the same direction, especially by Plotkin [Plo78] and Gunter [Gun87]. Droste hilighted the importance of having homogeneous universal objects for categories of domains, and obtained many results in this direction, using classical theorems of model theory [DG93]; he also did much work in the field of relating universal domains with their universal representations, mainly with event structures [Dro91] and Kahn and Plotkin's concrete data structures (see also [KP78]).

We firstly give a general, categorical introduction to universality, and express some results which we shall use in the following. Let $\mathcal{C}$ be a category where all arrows are monic, and $\mathcal{C}^*$ be a full subcategory of $\mathcal{C}$. An object $U \in \mathrm{Obj}(\mathcal{C})$ is

- $\mathcal{C}^*$-*universal* iff for every object $A \in \mathrm{Obj}(\mathcal{C}^*)$ there is an arrow $f : A \to U$; for example, in the case that $\mathcal{C}$ represents a preordered set, a $\mathcal{C}^*$-universal object represents an upper bound of $\mathcal{C}^*$;

- $\mathcal{C}^*$-*homogeneous* iff for any $A \in \mathrm{Obj}(\mathcal{C}^*)$ and for any two arrows $f, g : A \to U$ there exists an automorphism h of U (i.e., an arrow $h : U \to U$ which is an isomorphism of $\mathcal{C}$) such that $h \circ g = f$; in other words, every time that an object of $\mathcal{C}^*$ can be "mapped" into U using two arrows, these arrows just differ for the composition with an automorphism of U;

- $\mathcal{C}^*$-*saturated* iff for any $A, B \in \mathrm{Obj}(\mathcal{C}^*)$ and for any two arrows $f : A \to U$ and $g : A \to B$ there is an arrow $h : B \to U$ such that $h \circ g = f$; this can be interpreted as follows: if an object A of $\mathcal{C}^*$ can be mapped to U via f, and if it can also be mapped to some other object B, then f can be naturally extended to a map from B to U.

We are especially interested in the case when $\mathcal{C}$ is an algebroidal category (since all categories of representations are such) and $\mathcal{C}^*$ is the subcategory of all finite objects. For convenience of the reader, we recall here the basic definitions; a category is *semi-algebroidal* if every ω-chain of finite objects has a colimit, and moreover every object is the colimit of an ω-chain of finite objects. It is *algebroidal* if it is semi-algebroidal, the subcategory of finite objects has a countable skeleton, and for any two objects A, B the set $\mathrm{Hom}(A, B)$ is countable.

We then have the following important result, which is proved in [DG93]:

Theorem 4.2.1 (Droste and Göbel [DG93]) *Let $\mathcal{C}$ be an algebroidal category, U be an object of $\mathcal{C}$ and $\mathcal{C}_f$ be the full subcategory of its finite objects; then, the following are equivalent:*

1. *U is a $\mathcal{C}$-universal $\mathcal{C}_f$-homogeneous object;*

2. *U is a $\mathcal{C}_f$-saturated object.*

Moreover, if such statements hold, then U is unique up to isomorphism. $\square$

This gives us a way for proving that a certain object U is universal and homogeneous; we just have to show that, whenever we have an embedding $f : A \to U$ of a finite object A into it, and whenever A is embeddable into B, there is an embedding of B into U which simply extends f through the given embedding. So, one has to prove simply that it is possible to map every finite object into the (candidate) universal homogeneous one "step-by-step", and every time the embedding is simply obtained by extending the previous one.

What one usually wants, in order to simplify the proofs, is the possibility of proving the saturation property for the very special case that B is obtained from A by simply adding one element to it. In order to express this property in a categorical way, we define the concept of incremental category. An arrow $f : A \to B$ in an algebroidal category $\mathcal{C}$ is an *increment* iff $f = g \circ h$ implies that either g or h is an isomorphism. We say that $\mathcal{C}$ is incremental iff it contains a weakly initial object and for any morphism $f : A \to B$ between two finite objects $A, B \in \mathrm{Obj}(\mathcal{C}_f)$ there exists a finite chain $(A_i, f_i)_{i=0,\dots,n-1}$ such that $A = A_0$, $B = B_n$, $f = f_{n-1} \circ \cdots \circ f_1 \circ f_0$ and each $f_i : A_i \to A_{i+1}$ is an increment.

Using the same notations as before, we shall say that an object U of $\mathcal{C}$ is $\mathcal{C}^*$*-stepwise-saturated* iff for any two objects $A, B \in \mathrm{Obj}(\mathcal{C}^*)$ and any two arrows $f : A \to U$ and $g : A \to B$ such that g is an increment, there exists an arrow $h : B \to U$ with $h \circ g = f$. We prove the following:

Theorem 4.2.2 *Let $\mathcal{C}$ be an algebroidal incremental category and U be an object of $\mathcal{C}$. The following are equivalent:*

1. *U is a $\mathcal{C}$-universal $\mathcal{C}_f$-homogeneous object;*

2. *U is a $\mathcal{C}_f$-saturated object;*

3. *U is a $\mathcal{C}_f$-stepwise-saturated object.*

Proof: We just need to prove to second equivalence, the first one being just the statement of Theorem 4.2.1. It is clear that every saturated object is also stepwise saturated (in every category), so we prove the converse (which holds only in an incremental category). Suppose that $f : A \to U$ is an arrow from a finite object, and $g : A \to B$ is any arrow between two finite objects. Since the category is incremental, there exists a finite chain $(A =)A_0 \overset{g_0}{\to} A_1 \overset{g_1}{\to} \cdots A_{n-1} \overset{g_{n-1}}{\to} A_n(= B)$ where $g = g_{n-1} \circ \cdots \circ g_1 \circ g_0$ and each g_i is an increment. Let now $f_0 = f$; by definition of stepwise-saturation, we obtain a function $f_1 : A_1 \to U$ such that $f_1 \circ g_0 = f_0$, like in the commutative diagram

$$\begin{tikzcd}
A = A_0 \arrow[r, "g_0"] \arrow[d, "f=f_0"'] & A_1 \arrow[r, "g_1"] \arrow[dl, "f_1"] & A_2 & \cdots & A_{n-1} \arrow[r, "g_{n-1}"] & A_n = B \\
U & & & & &
\end{tikzcd}$$

Going on this way we finally obtain an arrow $f_n : B \to U$ with $f_n \circ g_{n-1} \circ \cdots g_1 \circ g_0 = f$. Letting $h = f_n$ we have $h \circ g = f$, as required. $\square$

4.3 Deterministic universality constructions

4.3.1 The category of tolerance spaces with embeddings

In this section, we shall mainly consider the category of tolerance spaces and its relations with the category of atomic coherent dI-domains. Tolerance spaces are also known from Girard [Gir87] as coherence (or coherent) spaces. A *tolerance space* (the term was introduced by Zeeman [Zee62], and was later studied in various kinds of forms, e.g. in [KKM90, Shr71]) $\mathcal{T} = (X, \mathrm{co})$ is a set X endowed with a binary reflexive symmetric relation co, which is often called the "consistency" or "indistinguishability" relation; the elements of X are sometimes called "nodes". From now on, we let X and co (possibly with suitable indexes) denote the underlying set and consistency relation associated to the tolerance space $\mathcal{T}$.

A *(tolerance)-continuous function* $f : \mathcal{T} \to \mathcal{T}'$ is a function between the underlying sets $f : X \to X'$ such that

$$\forall x, y \in X.\, x \,\mathrm{co}\, y \implies f(x) \,\mathrm{co}'\, f(y)$$

i.e., a homomorphism of tolerance spaces (a graph morphism, if we think of a tolerance space as a reflexive undirected graph). An *embedding*[1] is a homomorphism $f : \mathcal{T} \to \mathcal{T}'$ of tolerance spaces which is injective and such that

$$\forall x, y \in X.\, x \,\mathrm{co}\, y \iff f(x) \,\mathrm{co}'\, f(y).$$

Clearly, a tolerance space can be embedded into another one iff the former is an (induced) subgraph of the latter. If $\mathcal{T}$ is a tolerance space, and $Y \subseteq X$, we let $\mathcal{T}^Y = (Y, \mathrm{co} \cap Y^2)$ (i.e., the subgraph induced by Y). Clearly the inclusion map $i : \mathcal{T}^Y \to \mathcal{T}$ (where $i(y) = y$ for all $y \in Y$) is an embedding.
We let **TolSp** be the category of countable tolerance spaces with embedding as morphisms; the main properties of this category are listed in the following

Property 4.3.1 *In the category* **TolSp** *every arrow is monic, and an object is finite iff it has a finite underlying set. Moreover,* **TolSp** *is an algebroidal, incremental category.*

Proof: The fact that every arrow is monic follows because **TolSp** is a subcategory of **Set** containing only injective maps, and injective maps in **Set** are clearly monic.
Finite objects. First observe that, for every countable tolerance space $\mathcal{T}$, one can construct an ω-chain of finite tolerance spaces $(\mathcal{T}_i, f_i)$ with limit $\mathcal{T}$ (just take any sequence of finite sets $X_0 \subseteq X_1 \subseteq X_2 \ldots$ with union X and such that $|X_i| \leq i$ for every $i \in \omega$, and let $\mathcal{T}_i = \mathcal{T}^{X_i}$, with f_i defined taking the natural inclusion map). Since the presence of an embedding from $\mathcal{T}$ to $\mathcal{T}'$ obviously implies that $|X| \leq |X'|$, we immediately have that every finite object must have a finite support. For the converse, suppose that $\mathcal{T}$ is finite but have infinite support. Then, take the above ω-chain $(\mathcal{T}_i, f_i)$ whose limit is $\mathcal{T}$; since $\mathcal{T}$ can be embedded into itself using the identity, there must be an embedding of $\mathcal{T}$ into some finite set, contradicting the hypothesis. Note that this also proves that every object is the colimit of some ω-chain of finite objects.

[1]These morphisms are called "rigid embeddings" in [BCS93].

Limits of ω-chains. We must prove that every ω-chain of finite tolerance spaces has a limit. Without loss of generality, we can assume to have an ω-chain $(X_i, \mathrm{co}_i)_{i\in\omega}$ where $X_0 \subseteq X_1 \subseteq X_2 \ldots$. Now let $X = \cup_{i\in\omega} X_i$ and $\mathrm{co} = \cup_{i\in\omega}\mathrm{co}_i$. The tolerance space $\mathcal{T} = (X, \mathrm{co})$ is obviously the colimit of the above chain.
Algebroidality. At this point, we just need to prove that there are at most countably many non-isomorphic finite tolerance spaces (which is trivially true), and that there is only a countable number of embeddings between two finite tolerance spaces (in fact, there exists only a finite number of them).
Increments. Note that clearly the empty tolerance space is a weakly initial object. It is immediate to observe that an embedding $f : \mathcal{T} \to \mathcal{T}'$ between finite tolerance spaces is an increment precisely when $|X'| = |X| + 1$. It is then easy to see that every embedding can be decomposed into increments, adding one node at a time. □

A very important property related to the category of tolerance spaces was studied by Rado in [Rad67], where it is proved that a universal homogeneous tolerance space (more precisely: undirected graph) exists, and an explicit construction is given. Since in the next sections we shall provide various generalizations of this construction, we shall present it in full detail, giving the original definition of [Rad67] (for more on Rado's graph, see [Cam90]).

Theorem 4.3.1 (Rado [Rad67]) *Let $\mathcal{T}_{\mathrm{R}}$ be the tolerance space having ω as underlying set, and with compatibility relation co_{R} defined as follows*

$$n \ \mathrm{co}_{\mathrm{R}}\ m \iff \text{the } \min(n,m)\text{-th bit in the binary expansion of } \max(n,m) \text{ is “1”}.$$

$\mathcal{T}_{\mathrm{R}}$ is the universal homogeneous object of the category **TolSp***.*

Proof: By Property 4.3.1, using Theorem 4.2.2, we just have to prove that $\mathcal{T}_{\mathrm{R}}$ is stepwise saturated. We can limit ourselves, without loss of generality, to finite subspaces of $\mathcal{T}_{\mathrm{R}}$; so let $Y \subseteq_{\mathrm{fin}} \omega$ and suppose that $\mathcal{T} = (Y \cup \{*\}, \mathrm{co})$ is such that $\mathrm{co}^Y = \mathrm{co}_{\mathrm{R}}^Y$, where $* \notin \omega$. We have to find $x \in \omega \setminus Y$ in such a way that $\mathcal{T}$ is isomorphic to $\mathcal{T}_{\mathrm{R}}^{Y\cup\{x\}}$. Let $A = \{y \in Y : y \ \mathrm{co}\ *\}$, and define x as follows

$$x = 2^{1+\max Y} + \sum_{k\in A} 2^k.$$

We then must prove that, for every $y \in Y$ we have $x \ \mathrm{co}_{\mathrm{R}}\ y$ if and only if $* \ \mathrm{co}\ y$. Now, note that $x \geq 2^{1+\max Y}$ and so x is greater than every element of Y. So, for all $y \in Y$, $x \ \mathrm{co}_{\mathrm{R}}\ y$, if and only if the y-th bit in the binary expansion of x is 1, if and only if $y \in A$, which happens only when $* \ \mathrm{co}\ y$. □

4.3.2 A universal homogeneous coherent atomic dI-domain

The purpose of this section is to provide a proof of equivalence between the category of tolerance spaces with embeddings, and the category of coherent atomic dI-domains with stable embedding-projection pairs as morphisms. A consequence of this result is that we have a direct explicit construction of the universal homogeneous atomic coherent dI-domain (i.e., coherent qualitative domains).

We let **CAdIDom** be the category of countable coherent atomic dI-domains, with stable embedding-projection pairs as morphism. We first define a functor Clique from the category **TolSp** to **CAdIDom** as follows:

- for each tolerance space $\mathcal{T}$, we let Clique($\mathcal{T}$) be the poset whose elements are the cliques[2] of $\mathcal{T}$ (seen as an undirected graph); the order is given by inclusion;
- if $f : \mathcal{T}_0 \to \mathcal{T}_1$ is an embedding of tolerance spaces, we define Clique(f) $= \langle f_e, f_p \rangle$ where, for any two cliques $\sigma_0 \subseteq X_0$ and $\sigma_1 \subseteq X_1$, we let $f_e(\sigma_0) = \{f(x), x \in \sigma_0\} = f(\sigma_0)$ and $f_p(\sigma_1) = \{x \in X_0 : f(x) \in \sigma_1\} = f^{-1}(\sigma_1)$.

We shall prove that Clique gives an equivalence of categories, by using Theorem A.1.1. First, we must show that Clique is a well-defined functor.

Lemma 4.3.1 *Clique is a functor from the category* **TolSp** *to the category* **CAdIDom**.

Proof: We first prove that, for every tolerance space $\mathcal{T}$, Clique($\mathcal{T}$) is indeed an atomic coherent dI-domain. Let $\mathcal{D} =$ Clique($\mathcal{T}$) for a given tolerance space $\mathcal{T}$. Observe that $\mathcal{D}$ is really a poset, and least upper bounds, when they exist, are simply obtained as set-unions.

(i) *Completeness.* Let $S \subseteq D$ be directed, and $\sigma = \cup S$. If $x, y \in \sigma$ then $x \in \alpha$ and $y \in \beta$ for some $\alpha, \beta \in S$; hence, $\alpha \subseteq \gamma, \beta \subseteq \gamma$ for some $\gamma \in S$, and so $x, y \in \gamma$. But γ is a clique and so $x \operatorname{co} y$. Thus σ is a clique and $\sigma = \sqcup S$.

(ii) *Minimum element.* The minimum element is the empty clique.

(iii) *Compact elements.* We prove that a clique is compact if and only if it is finite. If $\alpha = \{x_1, \ldots, x_n\}$ is a finite clique and $\alpha \subseteq \cup S$ for some directed $S \subseteq D$, then $\exists \alpha_1, \ldots, \alpha_n \in S$ such that $x_i \in \alpha_i$ $(i = 1, \ldots, n)$. But, by directedness, there exists a $\beta \in S$ such that $\alpha_i \subseteq \beta$ for all $i = 1, \ldots, n$, and so $\alpha \subseteq \beta \in S$. Conversely, let α be isolated and infinite. Then we can find an infinite sequence of finite sets $\alpha_0 \subset \alpha_1 \subset \ldots$ such that $\cup_i \alpha_i = \alpha$. Each of them is a clique, and $S = \{\alpha_0, \alpha_1, \ldots\}$ is a directed set with $\alpha = \cup S$, but $\alpha \subseteq \alpha_i$ happens for no i.

(iv) *Algebraicity.* Let $\alpha \in D$; the set $K(\alpha)$ is the set of finite subcliques of α. Clearly $K(\alpha)$ is directed and its least upper bound is α. This also proves that $\mathcal{D}$ is finitary (a finite set has finitely many subsets).

(v) *Consistent completeness.* Straightforward.

(vi) *Distributivity.* Follows from set-theoretic distributivity, by simply noting that $\sqcap A = \cap A$ for every non-empty set A.

So far, we have shown that Clique($\mathcal{T}$) is a distributive Scott domain. Note that Clique($\mathcal{T}$) is also coherent; a pairwise compatible set of cliques is also compatible, and thus it has a least upper bound. For atomicity, clearly:

$$D^A = \{\{x\}, x \in X\}.$$

[2] As usual, a clique is simply a set $\sigma \subseteq X$ such that $x \operatorname{co} y$ whenever $x, y \in \sigma$.

and moreover, for every clique α, we have $K^A(\alpha) = \{\{x\}, x \in \alpha\}$ and so $\sqcup K^A(\alpha) = \cup_{x\in\alpha}\{x\} = \alpha$.

For the second part, we need to prove that Clique$(f) = \langle f_e, f_p\rangle$ is a stable embedding-projection pair, whenever $f : \mathcal{T}_0 \to \mathcal{T}_1$ is an embedding of tolerance spaces. First observe that, if σ_0, σ_1 are cliques of $\mathcal{T}_0, \mathcal{T}_1$, then $f_e(\sigma_0), f_p(\sigma_1)$ are cliques, because of the definition of embedding. Monotonicity and continuity follow from the fact that f_e and f_p are simply obtained as set-extensions of f and f^{-1}. For the property of stable embedding-projections we have the following; let σ_i be a clique of $\mathcal{T}_i$ (for $i = 0, 1$):

- since f is injective, one has $f_p(f_e(\sigma_0)) = f^{-1}(f(\sigma_0)) = \sigma_0$, as required;
- suppose now that $x \in f_e(f_p(\sigma_1))$; this means that $x = f(y)$ for some $y \in f^{-1}(\sigma_1)$, i.e., $x = f(y)$ and $f(y) \in \sigma_1$, which in turn implies $x \in \sigma_1$. So, $f_e(f_p(\sigma_1)) \subseteq \sigma_1$;
- for stability, suppose that $\sigma_1 \subseteq f_e(\sigma_0)$, i.e., $\sigma_1 \subseteq f(\sigma_0)$. Then
$$f_e(f_p(\sigma_1)) = f(f^{-1}(\sigma_1)).$$
If $x \in \sigma_1$, then $x \in f(\sigma_0)$, so $x = f(y)$ for some y. Thus, $y \in f^{-1}(\sigma_1)$, and so $x = f(y) \in f(f^{-1}(\sigma_1))$, as required.

So, Clique(f) is a stable embedding-projection pair. □

Moreover, every coherent atomic dI-domain is isomorphic to the domain of cliques of some tolerance space:

Lemma 4.3.2 *Let $\mathcal{D}$ be a coherent atomic dI-domain. There exists a tolerance space $\mathcal{T}$ such that Clique$(\mathcal{T}) \cong \mathcal{D}$.*

Proof: Let $X = D^\diamond$ and define $x \operatorname{co} y$ if and only if $x \uparrow y$. Now, consider this tolerance space, and define:
$$\begin{array}{rccc} \varphi : & \text{Clique}(X) & \to & D \\ & \alpha & \mapsto & \sqcup\alpha. \end{array}$$
Observe that, if α is a clique of X, then it is a pairwise compatible subset of D, and so $\sqcup\alpha$ exists. We must prove that φ is an order-isomorphism.

(i) *Injectivity.* Suppose $\varphi(\alpha) = \varphi(\beta)$, i.e. $\sqcup\alpha = \sqcup\beta$. Now $K^A(\sqcup\alpha) = K^A(\sqcup\beta)$ and so, using Lemma 3.4.1, $\alpha = \beta$.

(ii) *Surjectivity.* Straightforward, by using atomicity.

(iii) *Monotonicity.* Clearly, if $\alpha \subseteq \beta$ then $\sqcup\alpha \sqsubseteq \sqcup\beta$.

(iv) *Order preservation.* Suppose that $\sqcup\alpha \sqsubseteq \sqcup\beta$; then $K^A(\sqcup\alpha) = K^A(\sqcup\beta)$ and thus (using Lemma 3.4.1) $\alpha \subseteq \beta$. □

As a matter of fact, Clique gives a categorical equivalence, as explained in the following:

Theorem 4.3.2 *The functor Clique is a categorical equivalence between the categories* **TolSp** *and* **CAdIDom**.

Proof: Using Theorem A.1.1, by Lemmata 4.3.1 and 4.3.2, we just have to prove that Clique is full and faithful. For faithfulness, let $f, g : \mathcal{T}_0 \to \mathcal{T}_1$ be two embeddings such that $\mathrm{Clique}(f) = \mathrm{Clique}(g)$. Note that, for all $x \in X_0$, $f(x)$ is the only element of $f(\{x\}) = f_e(\{x\})$. Since $f_e = g_e$, we have $f = g$ as required.
The proof that Clique is full requires a more sophisticated technique. Suppose that $\mathcal{T}_0, \mathcal{T}_1$ are two tolerance spaces and $\langle f, g\rangle : \mathrm{Clique}(\mathcal{T}_0) \to \mathrm{Clique}(\mathcal{T}_1)$ is a stable embedding-projection pair. By Lemma 3.4.2, the image of every atom of $\mathrm{Clique}(\mathcal{T}_0)$ (i.e., of every singleton clique) is an atom of $\mathcal{T}_1$. This means that, for all $x \in X_0$, $f(\{x\})$ contains only one element of X_1. Define $h : X_0 \to X_1$ by letting $f(\{x\}) = \{h(x)\}$ (i.e., $h(x)$ is the only element of the set $f(\{x\})$). We first prove that h is an embedding of tolerance spaces. Injectivity follows from the injectivity of f.
Suppose now that $x \operatorname{co} y$ in $\mathcal{T}_0$. Then $\{x\}$ and $\{y\}$ are compatible in $\mathrm{Clique}(\mathcal{T}_0)$, and so (using Proposition 3.3.1) $f(\{x, y\}) = f(\{x\}) \cup f(\{y\}) = \{h(x)\} \cup \{h(y)\}$ which is a clique of $\mathcal{T}_1$, which means that $h(x) \operatorname{co} h(y)$. Conversely, suppose that $h(x) \operatorname{co} h(y)$, i.e., $\{h(x), h(y)\} \in \mathrm{Clique}(\mathcal{T}_1)$. By atomicity, $\{h(x)\} \cup \{h(y)\} \in \mathrm{Clique}(\mathcal{T}_1)$, i.e., $f(\{x\}) \operatorname{co} f(\{y\})$, which implies $x \operatorname{co} y$.
Finally, it is immediate to prove that $\langle f, g\rangle = \mathrm{Clique}(h)$. So Clique is full. □

As a corollary, we finally obtain:

Corollary 4.3.1 *Clique($\mathcal{T}_{\mathrm{R}}$) is the universal homogeneous coherent atomic dI-domain.*

Proof: By Theorem 4.3.1, using Theorem 4.3.2 and considering the Remark A.1.1. □

This gives a direct construction of the universal homogeneous coherent atomic dI-domain, which can be simply obtained as the domain of cliques of Rado's universal homogeneous graph.

4.3.3 The category of generalized tolerance spaces

In this section, we aim at generalizing the notion of tolerance space by substituting the tolerance relation (which is a binary one) with a more general kind of finitary predicate, following the line of [Gir87]. As a result, we shall obtain the category of what we call *generalized tolerance spaces*, for which we shall prove a generalization of Rado's theorem, using analogous number-theoretic constructions.

A *generalized tolerance space* (or gts for short) $\mathcal{T} = (X, \mathrm{Con})$ is a set X endowed with a predicate $\mathrm{Con} \subseteq \wp_{\mathrm{fin}}(X)$ satisfying the following conditions:

1. for all $x \in X$ we have $\{x\} \in \mathrm{Con}$;

2. Con is downward-closed, i.e., if $A \in \mathrm{Con}$ and $B \subseteq A$ then also $B \in \mathrm{Con}$.

The predicate Con can be interpreted as a consistency predicate, and we can think of a usual tolerance space as a special kind of generalized tolerance space where a finite set is consistent if and only if every pair of elements it contains is in the indistinguishability relation.

A *(gts-)continuous* function $f : \mathcal{T}_0 \to \mathcal{T}_1$ is a function $f : X_0 \to X_1$ such that, for all $A \in \mathrm{Con}_0$ we have $f(A) \in \mathrm{Con}_1$. In particular, an *embedding* (of generalized tolerance spaces) is an injective function $f : X_0 \to X_1$ such that, for all $A \in \wp_{\text{fin}}(X_0)$, it holds that $A \in \text{Con}_0$ iff $f(A) \in \text{Con}_1$.

If $Y \subseteq X$, we denote by $\mathcal{T}^Y$ the structure (Y, Con^Y), where $\text{Con}^Y = \text{Con} \cap \wp_{\text{fin}}(Y)$ (the proof that this is indeed a gts is easy, and omitted). Also in this case, the inclusion map $i : \mathcal{T}^Y \to \mathcal{T}$ is an embedding of generalized tolerance spaces.
We let **GTolSp** be the category of countable generalized tolerance spaces, with embeddings as arrows. Also in this case, we have that:

Property 4.3.2 *In the category* **GTolSp** *every arrow is monic, and an object is finite iff it has a finite underlying set. Moreover,* **GTolSp** *is an algebroidal, incremental category.*

Proof: The fact that every arrow is monic follows because **GTolSp** is a subcategory of **Set** containing only injective maps.
Finite objects. First observe that, for every countable gts $\mathcal{T}$, one can construct an ω-chain $(\mathcal{T}_i, f_i)$ with limit $\mathcal{T}$ (take any sequence of sets $X_0 \subseteq X_1 \subseteq X_2 \ldots$ with union X and such that $|X_i| \leq i$ for every $i \in \omega$, and let $\mathcal{T}_i = \mathcal{T}^{X_i}$, with f_i defined taking the natural inclusion map). Since the presence of an embedding from $\mathcal{T}$ to $\mathcal{T}'$ obviously implies that $|X| \leq |X'|$, we immediately have that every finite object must have a finite support. For the converse, suppose that $\mathcal{T}$ is finite but have infinite support. Then, take the above ω-chain $(\mathcal{T}_i, f_i)$ whose limit is $\mathcal{T}$; since $\mathcal{T}$ can be embedded into itself using the identity, there must be an embedding of $\mathcal{T}$ into some finite set, contradicting the hypothesis. Note that this also proves that every object is the colimit of some ω-chain of finite objects.
Limits of ω-chains. We must prove that every ω-chain of finite generalized tolerance spaces has a limit. Without loss of generality, take an ω-chain $(X_i, \text{Con}_i)_{i\in\omega}$ where $X_0 \subseteq X_1 \subseteq X_2 \ldots$. Now let $X = \cup_{i\in\omega} X_i$ and $\text{Con} = \cup_{i\in\omega}\text{Con}_i$. The generalized tolerance space $\mathcal{T} = (X, \text{Con})$ is obviously the colimit of the above chain.
Algebroidality. We are just left to show that there are at most countably many non-isomorphic finite gts's (which is trivially true), and that there is only a countable number of embeddings between two finite generalized tolerance spaces (in fact, there exists only a finite number of them).
Increments. Note that clearly the empty generalized tolerance space is a weakly initial object. It is immediate to observe that an embedding $f : \mathcal{T} \to \mathcal{T}'$ between finite gts's is an increment precisely when $|X'| = |X| + 1$. It is then easy to see that every embedding can be decomposed into increments, adding one element at a time. ☐

We are now ready to prove a generalization of Rado's theorem, holding for generalized tolerance spaces. We first need a way of coding finite sets of natural numbers, which we shall use in our construction:

Lemma 4.3.3 *There exist a (recursive) injection $\Psi : \wp_{\text{fin}}(\omega) \to \omega$ which satisfies the following:*

- *Ψ is not surjective;*
- *for all finite sets A of natural numbers, $\Psi(A)$ is an upper bound for A (in the usual ordering);*
- *if $A \subseteq B$ (where both sets are finite subsets of ω) then $\Psi(A) \leq \Psi(B)$.*

Proof: Let $\pi_1, \pi_2, \pi_3, \ldots$ be the sequence of prime numbers (i.e., $\pi_1 = 2$, $\pi_2 = 3$ etc.). Define Ψ as follows:

$$\Psi(\{n_1 < n_2 < \ldots < n_k\}) = \prod_{i=1}^{k} \pi_i^{n_i}.$$

The function is injective, by the Fundamental Theorem of Arithmetics, and obviously not surjective. Clearly, for every i, $n_i \leq \pi_i^{n_i} \leq \Psi(\{n_1 < n_2 < \ldots < n_k\})$, and so the first property is true. The second property is also straightforward. $\square$

Now, we define the (so far: candidate) universal homogeneous gts. Let $X_U = \omega$, and, for every $A \in \wp_{\mathrm{fin}}(X_U)$, let $A \in \mathrm{Con}_U$ iff the following constraints are satisfied:

1. $|A| \leq 1$ or
2. $|A| > 1$ and
   (a) for all $B \subset A$, $B \in \mathrm{Con}_U$;
   (b) the $\Psi(A \setminus \{\max A\})$-th bit in the unary expansion of $\max A$ is "1".

This is clearly a gts, which we shall denote by $\mathcal{T}_U$; now, using the technique of saturation, we shall prove that $\mathcal{T}_U$ is indeed the universal homogeneous gts.

Theorem 4.3.3 *$\mathcal{T}_U$ is the universal homogeneous object of the category* **GTolSp**.

Proof: By Property 4.3.2, using Theorem 4.2.2, we just have to prove that $\mathcal{T}_U$ is stepwise saturated. Suppose that $Y \subseteq_{\mathrm{fin}} \omega$ and that $\mathcal{T} = (Y \cup \{*\}, \mathrm{Con})$ is a gts with $\mathrm{Con}^Y = \mathrm{Con}_U^Y$. We want to find an $x \in \omega \setminus Y$ such that $\mathcal{T}$ is isomorphic to $\mathcal{T}_U^{Y \cup \{x\}}$. Now, let $\mathcal{C}$ be the set of elements A of Con^Y such that $A \cup \{*\} \in \mathrm{Con}$; clearly $\mathcal{C}$ is not empty (it contains the empty set) and is downward-closed. Now, define:

$$x = 2^{1+\Psi(Y)} + \sum_{A \in \mathcal{C}} 2^{\Psi(A)}.$$

We must prove that $\mathrm{Con}_U^{Y \cup \{x\}} = \mathrm{Con}_U^Y \cup \{A \cup \{x\} : A \in \mathcal{C}\}$.
For the left-to-right inclusion, suppose that $A \in \mathrm{Con}_U$, and consider $A \cap (Y \cup \{x\})$ which equals $A \cap Y$ if $x \notin A$, and $(A \cap Y) \cup \{x\}$ otherwise. There are two cases. If $x \notin A$, then $A \cap (Y \cup \{x\}) = A \cap Y \in \mathrm{Con}_U^Y$. Suppose, on the contrary, that $x \in A$: we shall prove that $A \cap Y \in \mathcal{C}$. Since A is consistent, every finite subset is. So, in particular $(A \cap Y) \cup \{x\}$ is consistent. Observing that $x \geq 2^{1+\Psi(Y)} > \Psi(Y)$, by Lemma 4.3.3 one gets that x is an upper bound for Y. So x is the maximum of $(A \cap Y) \cup \{x\}$. By consistency, one has that either $A \cap Y = \emptyset$ (which implies $A \cap Y \in \mathcal{C}$, because $\mathcal{C}$ is non-empty and downward closed) or that the $\Psi(A \cap Y)$-th bit of the binary expansion of x is a "1". But this would mean that either $\Psi(A \cap Y) = \Psi(Y) + 1$ (which is impossible, for the last item of Lemma 4.3.3), or $\Psi(A \cap Y) = \Psi(C)$ for some $C \in \mathcal{C}$, which in turn implies $A \cap Y \in \mathcal{C}$, by the injectivity of Ψ.
If $A \in \mathrm{Con}_U^Y$, then $A = C \cap Y$ for some $C \in \mathrm{Con}_U$. Observe that $x \geq 2^{1+\Psi(Y)} > \Psi(Y)$, so $x \notin Y$ and thus $x \notin A$. Take $D = C \setminus \{x\}$ (which is consistent, because C is). Then $D \cap (Y \cup \{x\}) = (D \cap Y) \cup (D \cap \{x\}) = D \cap Y$ which is in turn equal to $(C \cap \{x\}^C) \cap Y = (C \cap Y) \cap \{x\}^C = A \cap \{x\}^C = A$, so $A \in \mathrm{Con}_U^{Y \cup \{x\}}$.

Suppose now that $A \in \mathcal{C}$; we prove that $A \cup \{x\} \in \mathrm{Con}_U^{Y \cup \{x\}}$. The fact that $A \in \mathcal{C}$ implies that $A = C \cap Y$ for some $C \in \mathrm{Con}_U$, and thus also $A \in \mathrm{Con}_U$. We prove that $A \cup \{x\} \in \mathrm{Con}_U$ (and thus $(A \cup \{x\}) \cap (Y \cup \{x\}) = (A \cap Y) \cup (A \cap \{x\}) \cup (Y \cap \{x\}) \cup \{x\}$, which is $A \cup \{x\}$, will be an element of $\mathrm{Con}_U^{Y \cup \{x\}}$). In order to prove this, just consider the case $A \neq \emptyset$ (the other is obvious).

- If $B \subset A \cup \{x\}$, then either $x \notin B$ and so $B \in \mathrm{Con}_U$ (because $B \subseteq A$), or $B = D \cup \{x\}$ for some $D \subset A$. But then $x \geq 2^{\Psi(A)} > \Psi(A) > \Psi(D)$ and so $x = \max B$. Now, $D \in \mathcal{C}$ (since $D \subseteq A$) and thus the $\Psi(D)$-th bit of x is "1". So $B \in \mathrm{Con}_U$.

- Since $x = \max(A \cup \{x\})$, we should prove that the $\Psi(A)$-th bit of x is "1", which is true because $A \in \mathcal{C}$. $\square$

4.3.4 A universal homogeneous atomic dI-domain

The purpose of this section is to provide a proof of equivalence between the category of gts's with embeddings, and the category of atomic dI-domains with stable embedding-projection pairs as morphisms. A consequence of this result is that we shall have a direct explicit construction of the universal homogeneous atomic dI-domain (qualitative domain), by exactly mimicking what we did for the coherent case.

We let **AdIDom** be the category of countable atomic dI-domains, with stable embedding-projection pairs as morphism. We first define a functor Dom : **GTolSp** $\to$ **AdIDom** as follows:

- for each gts $\mathcal{T}$, we let $\mathrm{Dom}(\mathcal{T})$ be the poset whose elements are the consistent subsets of X, i.e., those sets $Y \subseteq X$ such that $\wp_{\mathrm{fin}}(Y) \subseteq \mathrm{Con}$; the order is given by inclusion;

- if $f : \mathcal{T}_0 \to \mathcal{T}_1$ is an embedding of gts's, we let $\mathrm{Dom}(f) = \langle f_e, f_p \rangle$ where, for any two consistent sets $Y_0 \subseteq X_0$ and $Y_1 \subseteq X_1$, we let $f_e(Y_0) = \{f(x), x \in Y_0\} = f(Y_0)$ and $f_p(Y_1) = \{x \in X_0 : f(x) \in Y_1\} = f^{-1}(Y_1)$.

We shall prove that Dom gives an equivalence of categories, by using Theorem A.1.1. First, we must show that Dom is a well-defined functor.

Lemma 4.3.4 *Dom is a functor from the category* **GTolSp** *to the category* **AdIDom**.

Proof: We first prove that, for every gts $\mathcal{T}$, $\mathrm{Dom}(\mathcal{T})$ is indeed an atomic dI-domain. Let $\mathcal{D} = \mathrm{Dom}(\mathcal{T})$ for a given gts $\mathcal{T}$. Observe that $\mathcal{D}$ is really a poset, and least upper bounds, when they exist, are simply obtained as set-unions.

(i) *Completeness.* Let $S \subseteq D$ be directed, and $Y = \cup S$. If $A \subseteq_{\mathrm{fin}} Y$ then, for each $x \in A$, there will be some $Y_x \in S$ such that $x \in Y_x$; hence, S being directed, there will be some $Y_A \in S$ such that $Y_x \subseteq Y_A$ for all $x \in A$. But Y_A is a consistent set, and so $A \subseteq \cup_{x \in A} Y_x \subseteq Y$ belongs to Con.

(ii) *Minimum element.* The minimum element is the empty consistent set.

(iii) *Compact elements.* We prove that a consistent set is compact if and only if it is finite. If $A = \{x_1, \ldots, x_n\}$ is a finite consistent set and $A \subseteq \cup S$ for some directed $S \subseteq D$, then $\exists Y_1, \ldots, Y_n \in S$ such that $x_i \in Y_i$ $(i = 1, \ldots, n)$. But, by directedness, there exists a $Y \in S$ such that $Y_i \subseteq Y$ for all $i = 1, \ldots, n$, and so $A \subseteq Y \in S$. Conversely, let Y be isolated and infinite. Then we can find an infinite sequence of finite sets $Y_0 \subset Y_1 \subset \ldots$ such that $\cup_i Y_i = Y$. Each of them is a consistent set, and $S = \{Y_0, Y_1, \ldots\}$ is a directed set with $Y = \cup S$, but $Y \subseteq Y_i$ happens for no i.

(iv) *Algebraicity.* Let $Y \in D$; the set $K(Y)$ is the set of finite consistent subsets of Y. Clearly $K(Y)$ is directed and its least upper bound is Y. This also proves that $\mathcal{D}$ is finitary (a finite set has finitely many subsets).

(v) *Consistent completeness.* Straightforward.

(vi) *Distributivity.* Follows from set-theoretic distributivity, by simply noting that $\sqcap A = \cap A$ for every non-empty set A.

So far, we have shown that Dom($\mathcal{T}$) is a distributive Scott domain. For atomicity, clearly:

$$D^A = \{\{x\}, x \in X\}.$$

and moreover, for every consistent set Y, we have $K^A(Y) = \{\{x\}, x \in Y\}$ and so $\sqcup K^A(Y) = \cup_{x \in Y}\{x\} = Y$.

For the second part, we need to prove that Clique(f) $= \langle f_e, f_p \rangle$ is a stable embedding-projection pair, whenever $f : \mathcal{T}_0 \to \mathcal{T}_1$ is an embedding of gts's. First observe that, if Y_0, Y_1 are consistent subsets of $\mathcal{T}_0, \mathcal{T}_1$, then $f_e(Y_0), f_p(Y_1)$ are also consistent, because of the definition of embedding. Monotonicity and continuity follow from the fact that f_e and f_p are simply obtained as set-extensions of f and f^{-1}. For the property of stable embedding-projections we have the following; let Y_i be a consistent subset of of $\mathcal{T}_i$ (for $i = 0, 1$):

- since f is injective, one has $f_p(f_e(Y_0)) = f^{-1}(f(Y_0)) = Y_0$, as required;
- suppose now that $x \in f_e(f_p(Y_1))$; this means that $x = f(y)$ for some $y \in f^{-1}(Y_1)$, i.e., $x = f(y)$ and $f(y) \in Y_1$, which in turn implies $x \in Y_1$. So, $f_e(f_p(Y_1)) \subseteq Y_1$;
- for stability, suppose that $Y_1 \subseteq f_e(Y_0)$, i.e., $Y_1 \subseteq f(Y_0)$. Then

  $$f_e(f_p(Y_1)) = f(f^{-1}(Y_1)).$$

  If $x \in Y_1$, then $x \in f(Y_0)$, so $x = f(y)$ for some y. Thus, $y \in f^{-1}(Y_1)$, and so $x = f(y) \in f(f^{-1}(Y_1))$, as required.

So, Dom(f) is a stable embedding-projection pair. □

Moreover, every atomic dI-domain is isomorphic to the domain of cliques of some generalized tolerance space:

Lemma 4.3.5 *Let $\mathcal{D}$ be an atomic dI-domain. There exists a generalized tolerance space $\mathcal{T}$ such that Dom($\mathcal{T}$) $\cong \mathcal{D}$.*

Proof: Let $X = D^\diamond$ and define $A \in \mathrm{Con}$ if and only if $A \uparrow$. Now, consider this tolerance space, and define:

$$\begin{array}{rccl} \varphi : & \mathrm{Dom}(X) & \to & D \\ & Y & \mapsto & \sqcup Y. \end{array}$$

Observe that, if Y is a consistent subset of X, then it is a pairwise compatible subset of D, and so $\sqcup Y$ exists. We must prove that φ is an order-isomorphism. Surjectivity of φ directly follows from atomicity. For injectivity, suppose $\varphi(Y) = \varphi(Y')$, i.e. $\sqcup Y = \sqcup Y'$. Now $K^A(\sqcup Y) = K^A(\sqcup Y')$ and so, using Lemma 3.4.1, $Y = Y'$. Clearly, if $Y \subseteq Y'$ then $\sqcup Y \sqsubseteq \sqcup Y'$, and so φ is monotone. For the converse, assume that $\sqcup Y \sqsubseteq \sqcup Y'$; then $K^A(\sqcup Y) = K^A(\sqcup Y')$ and thus (using Lemma 3.4.1) $Y \subseteq Y'$. □

Once more, Dom gives a categorical equivalence, as explained in the following:

Theorem 4.3.4 *The functor Dom is a categorical equivalence between the categories of* **GTolSp** *and* **AdIDom***.*

Proof: Using Theorem A.1.1, by Lemmata 4.3.1 and 4.3.5, we just have to prove that Dom is full and faithful. For faithfulness, let $f, g : \mathcal{T}_0 \to \mathcal{T}_1$ be two embeddings such that $\mathrm{Dom}(f) = \mathrm{Dom}(g)$. Note that, for all $x \in X_0$, $f(x)$ is the only element of $f(\{x\}) = f_e(\{x\})$. Since $f_e = g_e$, we have $f = g$ as required.
The proof that Dom is full is essentially the same as in Theorem 4.3.2, and therefore omitted. □

As a corollary, we finally obtain:

Corollary 4.3.2 *$\mathrm{Dom}(\mathcal{T}_U)$ is the universal homogeneous atomic dI-domain.*

Proof: By Theorem 4.3.3, using Theorem 4.3.4 and considering the Remark A.1.1. □

This gives a direct construction of the universal homogeneous atomic dI-domain, which can be simply obtained as the domain of cliques of the generalized universal homogeneous gts described in the previous section.

4.3.5 A universal construction for event structures having minimum enabling

The concepts of tolerance space and generalized tolerance space both deal with the problem of defining "consistency": in the first case (in)consistency is given by a binary relation (and the only consistent sets are those which are cliques of this basic relation), in the second case it is given by a finitary predicate. It is not difficult to see that these are really special cases of a much more general kind of structure, where, besides consistency, enabling is also taken into consideration. These structures, introduced in [Win80], are a well-studied subject of domain theory, which finds many applications both in denotational semantics [Dro91, Dro89] (with the aim of obtaining representation theorems for various classes of domains), and in the theory of concurrency [NPW81, RT91, Win87, BCS93] (where event structures are a paradigmatic way of representing truly concurrent processes). We shall later on discuss in more detail the notion of event structure, taking into considerations different variants of this notion.

For the purpose of this section, we shall simply introduce the definition of general event structure and that of event structure with minimum enabling, and provide a universal homogeneous object for the latter category. We postpone to a later section the problem of whether this gives also a universal object for the corresponding domain category or not.

An *event structure* [Win80] is a triple $\mathcal{E} = (E, \mathrm{Con}, \vdash)$ where (E, Con) is a generalized tolerance space[3] (the elements of E are called "events"), $\vdash \subseteq \mathrm{Con} \times E$ is the enabling relation satisfying the following "inheritance condition":

$$A \vdash e, A \subseteq B \in \mathrm{Con} \implies B \vdash e.$$

A set $X \subseteq E$ is called a *configuration* (or state) of $\mathcal{E}$ iff

1. X is a consistent set, i.e., $\wp_{\mathrm{fin}}(X) \subseteq \mathrm{Con}$;
2. for all $x \in X$ there exists a *securing chain*, i.e., a sequence $x_1, x_2, \ldots, x_n = x \in X$ such that, for every $i = 1, \ldots, n$, one has $\{x_j, 1 \leq j < i\} \vdash x_i$.

The set of configurations of $\mathcal{E}$ is denoted by $\mathcal{L}(\mathcal{E})$. We use the same symbol to denote the corresponding poset w.r.t. set-theoretic inclusion[4].

An event structure $\mathcal{E}$ has the *minimum enabling property* (or, shortly, it is a MeES) iff for every $e \in E$ one of the following happens:

1. either no $A \in \mathrm{Con}$ is such that $A \vdash e$;
2. or else, if we put $\mu_e = \cap_{A \in \mathrm{Con}, A \vdash e} A$, it happens that $\mu_e \vdash e$ (μ_e is called the minimum enabling for e); said otherwise, the set $\{A \in \mathrm{Con} : A \vdash e\}$ has a minimum element μ_e.

In other words, either e cannot be enabled in any way, or otherwise there is only one minimum enabling for it (every other enabling set contains that minimum enabling). This definition resembles that of stable event structures, but it is not exactly the same, even though they completely coincides as far as we look at the configuration domains. Rather, it is similar to the definition of prime event structure [BCS93, NPW81], where the minimum enabling is simply the principal ideal generated by the given event in the causality ordering.

In this section, we shall further generalize our constructions and obtain in this way a universal homogeneous event structure with minimum enabling. For this purpose, we define what is an event structure homomorphism and embedding. A *homomorphism* of event structures $f : \mathcal{E}_0 \to \mathcal{E}_1$ is simply a gts-continuous function $f : (E_0, \mathrm{Con}_0) \to (E_1, \mathrm{Con}_1)$ such that moreover

$$\forall A \in \mathrm{Con}_0 \forall e \in E_0.\, A \vdash_0 e \implies f(A) \vdash_1 f(e).$$

[3] To be precise, this is not exactly the definition given in [Win80], because here we are requiring that every singleton is consistent. Of course, this does not harm when considering configurations, because events which are not themselves consistent will simply never appear in a configuration.

[4] We postpone the discussion of which kinds of domains can be generated as configuration posets of event structures; for the time being, we are just interested in their "syntactic" structure.

We say that f is an *embedding* iff it is an embedding of the underlying generalized tolerance spaces and moreover

$$\forall A \in \mathrm{Con}_0 \forall e \in E_0.\, A \vdash_0 e \iff f(A) \vdash_1 f(e).$$

The category of countable MeES's with embeddings will be denoted by **MeES**.

Given a MeES $\mathcal{E}$ and a subset $Y \subseteq E$, define $\mathcal{E}^Y$ to be the structure $(Y, \mathrm{Con}^Y, \vdash^Y)$, where $\mathrm{Con}^Y = \mathrm{Con} \cap \wp_{\mathrm{fin}}(Y)$ and $\vdash^Y = \vdash \cap(\mathrm{Con}^Y \times Y)$. In this case, a little thought is required to get convinced that $\mathcal{E}^Y$ is still a MeES:

Lemma 4.3.6 *Let $\mathcal{E}$ be a MeES, and $Y \subseteq E$. Then $\mathcal{E}^Y$ is a MeES, and if $e \in Y$ has an enabling in $\mathcal{E}^Y$ then $\mu_e^Y = \mu_e$.*

Proof: The only non-trivial part is the existence of the minimum enabling. By contradiction, suppose that $A, B \in \mathrm{Con}^Y$ are such that $A \vdash^Y e$ and $B \vdash^Y e$, both in the minimal way (i.e., no proper subset of A or B enable e), and $A \neq B$. Then also $A \vdash e$ and $B \vdash e$, so $\mu_e \subseteq A \cap B \subseteq Y$, thus also $\mu_e \vdash^Y e$, which contradicts the fact that A and B enable e in a minimal way. □

This lemma serves to prove that **MeES** is an incremental category, with one-point extensions. Note that, as in the case of (generalized) tolerance spaces, the inclusion morphism is an embedding of MeES's.

The main property of the category **MeES** are listed in the following:

Property 4.3.3 *In the category* **MeES** *every arrow is monic, and an object is finite iff it has a finite event set. Moreover,* **MeES** *is an algebroidal, incremental category.*

Proof: The proof is analogous to that of Properties 4.3.1 and 4.3.2, using Lemma 4.3.6 for incrementality. □

We shall now build up the universal homogeneous MeES using the same number-theoretical techniques developed in the previous sections. Build the event structure $\mathcal{E}_U$ by letting $E_U = \omega$, and Con_U, $\vdash_U$ defined as follows:

- a finite set $A \subseteq \omega$ belongs to Con_U if and only if
    1. $|A| \leq 1$ or
    2. $|A| > 1$ and
        (a) for all $B \subset A$, $B \in \mathrm{Con}_U$;
        (b) the $2\Psi(A \setminus \{\max A\})$-th bit in the unary expansion of $\max A$ is "1";
- $A \vdash_U e$ if and only if $A \in \mathrm{Con}_U$ and, letting

$$k_e = \min(\{k : \text{ the } (2k+1)\text{-th bit in the binary expansion of } e \text{ is "1"}\} \cup \{0\}),$$

  we have that $\Psi^{-1}(k_e) = \{B\}$ and $B \subseteq A$.

The essential idea behind this definition is that each event (number) encodes in its binary expansion all the information about consistency and enabling; in particular, the even-position bits are used to encode consistency (just in the same way as we did for gts's), while the first odd-position bit is used to encode the minimum enabling set. Note that the other odd-position bits can be freely set to "1": this is needed in order to prove saturation, because we need infinitely many events having the same consistency and enabling properties to be able to add them at every possible step of the embedding in $\mathcal{E}_U$. Moreover, note that, if no "1" in the binary expansion of e has odd position, or if the first odd-indexed "1" corresponds to a position which does not code a unique set, then we assume that no set ever enables e.

Observe that:

Lemma 4.3.7 *The structure $\mathcal{E}_U$ is an event structure with minimum enabling.*

Proof: Only minimum enabling needs a proof. Suppose that there is a set $A \in \mathrm{Con}_U$ such that $A \vdash_U e$. This means that $\Psi^{-1}(k_e) = \{B\}$ with $B \subseteq A$. But then $B \in \mathrm{Con}_U$ and clearly $B \vdash_U e$. Moreover:

$$A \in \mathrm{Con}_U \ \wedge \ A \vdash_U e \implies B \subseteq A.$$

So $\mu_e = \bigcap_{A \in \mathrm{Con}_U, A \vdash_U e} A = B \vdash_U e$ as required. □

We are now ready to prove the universality statement:

Theorem 4.3.5 *$\mathcal{E}_U$ is the universal homogeneous object of the category* **MeES**.

Proof: By Property 4.3.3, using Theorem 4.2.2, we just have to prove that $\mathcal{E}_U$ is stepwise saturated. This can be restated as follows: let $Y \subseteq_{\text{fin}} E_U$, $A \in \mathrm{Con}_U^Y$ and $\mathcal{C} \subseteq \mathrm{Con}_U^Y$ be non empty; then, we must find two elements $x, y \in X_U \setminus Y$ such that

$$\mathrm{Con}_U^{Y \cup \{x\}} = \mathrm{Con}_U^Y \cup \{B \cup \{x\}, B \in \mathcal{C}\}$$

and the same holds for y, and moreover

$$\begin{aligned}
\vdash_U^{Y \cup \{x\}} &= \vdash_U^Y \cup \{(B \cup \{x\}, e) : e \in Y, B \vdash_U^Y e, B \cup \{x\} \in \mathrm{Con}_U^{Y \cup \{x\}}\} \\
\vdash_U^{Y \cup \{y\}} &= \vdash_U^Y \cup \{(B, y) : A \subseteq B \in \mathrm{Con}_U^{Y \cup \{y\}}\} \cup \\
& \quad \cup \{(B \cup \{y\}, e) : e \in Y, B \vdash_U^Y e, B \cup \{y\} \in \mathrm{Con}_U^{Y \cup \{y\}}\}.
\end{aligned}$$

Before going on in this proof, we must explain why this is equivalent to proving the stepwise-saturation property. The idea is that $\mathcal{C}$ represents the set of consistent sets with which we want to mantain the consistency by adding the new event, while A represents the minimum enabling set for the new event. As a matter of fact, this holds for y, while x is a new event with no enabling at all, and we want to be sure that we can still find it.

Existence of y. Take

$$y = 2^{2+2\Psi(Y)} + \sum_{B \in \mathcal{C}} 2^{2\Psi(B)} + 2^{1+2\Psi(A)}.$$

We prove separately the two parts relative to consistency and enabling.

Consistency: left-to-right inclusion. Suppose $B \in \mathrm{Con}_U^{Y \cup \{y\}}$. If $B \subseteq Y$, then $B \in \mathrm{Con}_U^Y$. (Note that $Y \geq 2^{2+2\Psi(Y)} > \Psi(Y)$ and so $y \notin Y$). Suppose now that $y \in B$. Then consistency of B implies that (since $B \setminus \{y\} \subseteq Y \implies y = \max B$) the $2\Psi(B \setminus \{y\})$-th bit of y is "1". This means that $B \setminus \{y\} \in \mathcal{C}$ (by definition of y) and so $B \in \{C \cup \{y\}, C \in \mathcal{C}\}$, as required.

Consistency: right-to-left inclusion. Clearly, $\mathrm{Con}_U^Y = \mathrm{Con}_U \cap \wp_{\mathrm{fin}}(y) \subseteq \mathrm{Con}_U \cap \wp_{\mathrm{fin}}(Y \cup \{y\}) = \mathrm{Con}_U^{Y \cup \{y\}}$. Now suppose $B \in \mathcal{C}$: we prove that $B \cup \{y\} \in \mathrm{Con}_U$. But $B \in \mathcal{C} \implies B \subseteq Y \implies y = \max(B \cup \{y\})$. So, to prove $B \cup \{y\} \in \mathrm{Con}_U$, we have:

1. if $B = \emptyset$, the result holds trivially;

2. otherwise, we first have to check that the $\Psi(B)$-th bit of y is "1" (which is true, becasue $B \in \mathcal{C}$); then, let $C \subseteq B$. To check $C \in \mathrm{Con}_U$, suppose $y \in C$ (the other case is trivial, since $B \in \mathrm{Con}_U$). The result is then obvious for similar reasons.

Enabling: left-to-right inclusion. Suppose $B \vdash_U^{Y \cup \{y\}} e$. If $B \subseteq Y$ and $e \neq y$, then also $B \vdash_U^Y e$. For the other cases:

1. suppose $e = y$; then $B \vdash_U^{Y \cup \{y\}} y$, and so $B \vdash_U y$, hence $\Psi^{-1}(k_y) = \{C\}$ and $C \subseteq B$. But $k_y = \Psi(A)$, so $\Psi^{-1}(k_y) = \Psi^{-1}(\Psi(A)) = \{A\}$ and thus $A \subseteq B$, as required.

2. the remaining case is when $y \in B$ and $e \neq y$. So $B = C \cup \{y\}$ (with $C \subseteq Y$) and $B \vdash_U^{Y \cup \{y\}} e \neq y$. Suppose that $\neg(C \vdash_U^Y e)$, and let $\Psi^{-1}(k_e) = \{D\}$. Clearly $D \subseteq B$ but $D \nsubseteq C$, so necessarily $y \in D$. Thus $k_e = \Psi(D) \geq y$. So in the binary expansion of e, there is a bit "1" in a position at least $2y+1$. Thus $e \geq 2^{1+2y}$. Yet $e \in Y$ and so $e \leq \Psi(Y)$; so necessarily $2^{1+2y} \leq \Psi(Y)$. But $y \geq 2^{2+2\Psi(Y)}$ so $2^{1+2y} \geq 2^{1+2^{3+2\Psi(Y)}}$ which is greater than $\Psi(Y)$. Thus, by contradiction, we have $C \vdash_U^Y e$.

Enabling: right-to-left inclusion. Note that $\vdash_U^Y$ equals $\vdash_U \cap (\wp_{\mathrm{fin}}(Y) \times Y)$ which is included in $\vdash_U \cap (\wp_{\mathrm{fin}}(Y \cup \{y\}) \times (Y \cup \{y\}))$ which equals $\vdash_U^{Y \cup \{y\}}$. For the second case, suppose that $B \in \mathrm{Con}_U^{Y \cup \{y\}}$ and $A \subseteq B$. We must prove that $B \vdash_U^{Y \cup \{y\}} y$, i.e., that $B \vdash_U y$. To do this, we must have that $\Psi^{-1}(k_y) = \{D\}$ with $D \subseteq B$. But $k_y = \Psi(A)$, so $\Psi^{-1}(k_y) = \{A\}$, and $A \subseteq B$ by hypothesis. For the last case, suppose that $B \vdash_U^Y e$ and $B \cup \{y\} \in \mathrm{Con}_U^{Y \cup \{y\}}$. Then $B \cup \{y\} \vdash_U^{Y \cup \{y\}} e$ as required.

Existence of x. Let $q \notin \Psi(\wp_{\mathrm{fin}}(A))$ (this exists, because Ψ is not surjective). Define:

$$x = 2^{2+2\Psi(Y)} + \sum_{B \in \mathcal{C}} 2^{2\Psi(B)} + 2^{1+2q}.$$

The proof for consistency is exactly the same as for y, and omitted.

Enabling: left-to-right inclusion. Suppose $B \vdash_U^{Y \cup \{x\}} e$. If $B \subseteq Y$ and $e \neq x$, then $B \vdash_U^Y e$. So, consider the other cases:

1. if $e = x$, then $B \vdash_U x$ which implies that $\Psi^{-1}(k_x) = \{D\}$ with $D \subseteq B$. But $k_x = q$ and $\Psi^{-1}(q) = \emptyset$ by definition;

2. suppose that $x \in B$ and $e \neq x$. Let $C = B \backslash \{x\} \in \mathrm{Con}_U^Y$. We must show that $C \vdash_U e$. Suppose that $\neg(C \vdash_U e)$ but $B \vdash_U e$. Then $\Psi^{-1}(k_e) = \{D\}$ with $D \subseteq B$ but $D \not\subseteq C$. So $x \in D$, and thus $k_e = \Psi(D) \geq x$. So, in the binary expansion of e there is a bit "1" in a position which is at least $2x+1$, i.e., $e \geq 2^{1+2x}$. But $e \in Y$ and so $e \geq \Psi(Y)$; so, necessarily, $2^{1+2x} \geq \Psi(Y)$. But $x \geq 2^{2+2\Psi(Y)}$ so $2^{1+2x} \geq 2^{1+2^{3+2\Psi(Y)}}$ which is greater that $\Psi(Y)$. Thus, by contradiction, we have $C \vdash_U^Y e$.

Enabling: right-to-left inclusion. It is straightforward to check that $\vdash_U^Y \subseteq \vdash_U^{Y \cup \{x\}}$. Now, suppose that $B \vdash^Y e$, with $e \in Y$ and $B \cup \{x\} \in \mathrm{Con}_U^{Y \cup \{x\}}$. Then also $B \cup \{x\} \vdash_U^{Y \cup \{x\}} e$ as required. $\square$

A question arises naturally, whether we can use this construction also for obtaining a universal homogeneous domain of some kind, by proving an equivalence between the category **MeES** and a domain category. We shall give a (negative) answer to this question in the next section.

4.3.6 MeES's and the category of dI-domains

In this section, we shall provide a representation theorem for the class of dI-domains, by proving that the domain of configurations of a MeES is always a dI-domain, and conversely every dI-domain can be obtained as the domain of configurations of such a structure. The proof is quite standard, and similar, for the techniques used, to those presented for example in [Win80] or [NPW81].

We first prove the easy part:

Theorem 4.3.6 *For every MeES $\mathcal{E}$, the poset $\mathcal{L}(\mathcal{E})$ (w.r.t. inclusion ordering) is a dI-domain.*

Proof: (First part) Let $\mathcal{D} = \mathcal{L}(\mathcal{E})$; we first prove that $\mathcal{D}$ is a cpo. Clearly $\mathcal{D}$ contains a minimum element, i.e., the empty configuration $\emptyset$. Suppose now that $S \subseteq D$ is a directed set of configurations, and let $X = \cup S$: we must show that X is also a configuration. If $A \subseteq_{\mathrm{fin}} X$, say $A = \{x_1, \ldots, x_n\}$, then each x_i is included in some configuration of S: by directedness of S, there must be some $Y \in S$ such that $A \subseteq Y$, and so $A \in \mathrm{Con}$. Now, suppose that $e \in X$; then $e \in Y$ for some $Y \in S$, and so Y contains a securing chain $e_1, e_2, \ldots, e_n = e$ for e, and this is also a securing chain for e in X.
For consistent completeness, let $S \subseteq D$ be an upper-bounded set of configurations: this means that there exists a configuration X such that $\cup S \subseteq X$. Now, let $Y = \cup S$: we shall prove that Y is a configuration, which is clearly the least upper bound for Y. Every finite subset of Y is also a finite subset of X, and so it must be consistent. Moreover, if $e \in Y$, then e belongs to some configuration of S, and the same configuration must therefore contain a securing chain for e: this is also a securing chain for e in Y. Thus, finally, $Y \in \mathcal{D}$.
Now, we want to prove that a configuration is compact if and only if it is finite. It is immediate to see that every finite configuration is compact. For the converse, suppose that X is compact; we build a sequence $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$ of subsets of X as follows:

- put $X_0 = \emptyset$;

- if $X = X_k$ then let $X_{k+1} = X_k$; otherwise, choose an element $e \in X \setminus X_k$, and let $e_1, \ldots, e_n = e$ be a securing chain of e in X. Let $X_{k+1} = X_k \cup \{e_1, \ldots, e_n\}$.

Note that each X_i is a finite configuration, and moreover at each step the cardinality is strictly increased (unless X is finite, in which case the sequence constantly equals X from a certain index on). Since $X = \cup_{i\in\omega} X_i$, by compactness of X, we must have $X \subseteq X_i$, and so X must be finite.
For each configuration X, the set $K(X)$ is directed (the union of two finite compatible configurations is a finite configuration), and its least upper bound is X (every event appears in some finite subconfiguration of X). So ω-algebraicity is guaranteed. Also, $\mathcal{D}$ is finitary, because a finite configuration can have only finitely many subconfigurations (in fact: subsets).

So far, we have proved that $\mathcal{D}$ is a finitary Scott domain, where least upper bounds of directed (compatible) sets are obtained simply as unions, and the compact elements are just the finite configurations.

We still have to prove distributivity. Before proceeding in the proof, we need some technical definitions and lemmata. In the rest of this subsection, we assume that $\mathcal{E}$ is a MeES.
A safe sequence for an event $e \in E$ is a sequence $\gamma = (e_1, e_2, \ldots, e_k)$ where $e_k = e$ and, for all $i = 1, \ldots, k$ it holds that $\{e_1, \ldots, e_{i-1}\} \vdash e_i$. We let $\mathrm{ev}(\gamma) = \{e_1, e_2, \ldots, e_k\}$ be the set of events occurring in the sequence γ. If $A \subseteq E$ and $\gamma = (e_1, e_2, \ldots, e_k)$ is a safe sequence, we let $\gamma \upharpoonright A$ be the subsequence of γ built by taking the only elements of γ which belong to A, in their order.

Lemma 4.3.8 *Let Γ be a set of safe sequences for an event $e \in E$, and let* $\mathrm{ev}(\Gamma) = \cap_{\gamma\in\Gamma}\mathrm{ev}(\gamma)$. *For each $\gamma \in \Gamma$, the sequence $\gamma \upharpoonright \mathrm{ev}(\Gamma)$ is a safe sequence for e.*

Proof: Suppose that $\gamma = (e_1, e_2, \ldots, e_k = e)$; since $e \in \mathrm{ev}(\gamma)$ for all $\gamma \in \Gamma$, we have that $e \in \mathrm{ev}(\Gamma)$. So $\gamma \upharpoonright \mathrm{ev}(\Gamma) = (e'_1, \ldots, e'_h)$ will be such that $e'_h = e$. Now, we must prove that $\{e'_1, \ldots, e'_{i-1}\} \vdash e'_i$. Clearly, since $e'_i \in \mathrm{ev}(\gamma)$, every sequence δ of Γ contains e'_i somewhere, and so there is a subset of its event set enabling e'_i, i.e., there is some $x_\delta \subseteq \mathrm{ev}(\delta)$ such that $x_\delta \vdash e'_i$. Consider $x = \cap_{\delta\in\Gamma} x_\delta$: this is clearly a subset of $\mathrm{ev}(\Gamma)$, and moreover $\mu_{e'_i} \subseteq x$: so, $x \vdash e'_i$. But $\{e'_1, \ldots, e'_{i-1}\} = x$ and so we are done. □

We can thus obtain the following:

Corollary 4.3.3 *Let $e \in E$ be an event such that there is a configuration $X \in \mathcal{L}(\mathcal{E})$ with $e \in X$. Define:*

$$\lceil e \rceil = \cap\{X \in \mathcal{L}(\mathcal{E}) : e \in X\}.$$

We have that $\lceil e \rceil \in \mathcal{L}(\mathcal{E})$.

Proof: Consistency is straightforward. For enabling, let $e' \in \lceil e \rceil$. Since $e' \in X$ for all $X \in \mathcal{L}(\mathcal{E})$ with $e \in X$, e' has a safe sequence in X. Using the result of Lemma 4.3.8, we can find a securing sequence for e' also in $\lceil e \rceil$ (by taking Γ to be the set of all safe sequences corresponding to the various X in the set). □

Moreover:

Lemma 4.3.9 *The set of complete primes of $\mathcal{L}(\mathcal{E})$ is precisely the set of configurations of the form*

$$\lceil e \rceil = \cap\{X \in \mathcal{L}(\mathcal{E}) : e \in X\}$$

where the latter set is non-empty.

Proof: Suppose first that $S \subseteq \mathcal{L}(\mathcal{E})$ is a compatible set and $\lceil e \rceil \subseteq \cup S$. Then $e \in \lceil e \rceil \subseteq \cup S$, so $e \in X$ for some $X \in S$. But then also $\lceil e \rceil \subseteq X \in S$.
For the converse, suppose that X is a complete prime. Take the set $A = \{\lceil e \rceil, e \in X\}$: this is a compatible set of configurations, and $X = \cup A$. But then $X \subseteq \lceil e \rceil$ for some $e \in X$, and so $X = \lceil e \rceil$ for some event e. $\square$

So, finally, we obtain that:

Proof of Theorem 4.3.6 (continued): If X is a configuration, then clearly $X = \cup_{e \in X} \lceil e \rceil$. So $\mathcal{L}(\mathcal{E})$ is prime algebraic and thus, by Theorem 3.4.1, $\mathcal{L}(\mathcal{E})$ is a dI-domain. $\square$

Now, the converse of Theorem 4.3.6 can be obtained by taking into consideration the class of prime event structures, in the sense of [Win87]. A (generalized[5]) prime event structure (Winskel [Win87]) is a structure $\mathcal{E} = (E, \mathrm{Con}, \leq)$ where E and Con are like in the definition of a MeES, and moreover $\leq$ is a partial order on E such that:

1. for all $e \in E$, the set $\downarrow e = \{e' \in E : e' \leq e\}$ is finite;

2. if $X \in \mathrm{Con}$ and there exists $e' \in X$ such that $e \leq e'$ then $X \cup \{e\} \in \mathrm{Con}$.

We let $\mathcal{L}^{\mathrm{PR}}(\mathcal{E})$ be the set of subsets $X \subseteq E$ such that $\wp_{\mathrm{fin}}(X) \subseteq \mathrm{Con}$ and $\downarrow e \subseteq X$ for all $e \in X$. The elements of $\mathcal{L}^{\mathrm{PR}}(\mathcal{E})$ are called the configurations of the prime event structure $\mathcal{E}$. It is known that:

Theorem 4.3.7 (Winskel [Win87]) *For every dI-domain $\mathcal{D}$, there exists a (generalized) prime event structure $\mathcal{E}^{\mathrm{PR}}(\mathcal{D})$ such that $\mathcal{L}^{\mathrm{PR}}(\mathcal{E}^{\mathrm{PR}}(\mathcal{D})) \cong \mathcal{D}$.* $\square$

So, now we have:

Theorem 4.3.8 *For every dI-domain $\mathcal{D}$, there is a MeES $\mathcal{E}(\mathcal{D})$ such that $\mathcal{L}(\mathcal{E}(\mathcal{D})) \cong \mathcal{D}$.*

Proof: Consider the generalized prime event structure $\mathcal{E}^{\mathrm{PR}}(\mathcal{D}) = (E, \mathrm{Con}, \leq)$ obtained as in Theorem 4.3.7. Now, for each $X \in \mathrm{Con}$ and $e \in E$, define $X \vdash e$ if and only if $\downarrow e \setminus \{e\} \subseteq X$; let $\mathcal{E}(\mathcal{D}) = (E, \mathrm{Con}, \vdash)$. We first prove that $\mathcal{E}(\mathcal{D})$ is a MeES. Suppose that e is somehow enabled: then, there is $X \in \mathrm{Con}$ with $\downarrow e \subseteq X$, but thus also $\downarrow e \in \mathrm{Con}$, and thus clearly $\downarrow e = (\cap_{X \vdash e} X) \cup \{e\}$. So $\mu_e = \downarrow e \setminus \{e\} \vdash e$, as required.
Now, it is immediate to verify that $\mathcal{L}(\mathcal{E}(\mathcal{D})) = \mathcal{L}^{\mathrm{PR}}(\mathcal{E}^{\mathrm{PR}}(\mathcal{D})) \cong \mathcal{D}$. $\square$

To summarize, we can say that the domains of configurations of MeES's are precisely (up to isomorphism) all and only the dI-domains. In other words, if **dIDom** is the category

[5]Later on, we will have to do with other kinds of prime event structures, with binary conflict.

of (countable) dI-domains, with stable embedding-projection pairs as morphisms, we have a map

$$\mathcal{L} : \mathrm{Obj}(\mathbf{MeES}) \to \mathrm{Obj}(\mathbf{dIDom})$$

with the property that every object of **dIDom** is isomorphic to some object in the image of $\mathcal{L}$.

In order to prove (or disprove) the categorical equivalence, we must extend $\mathcal{L}$ to a functor: we do this as follows. Suppose that $f : \mathcal{E}_0 \to \mathcal{E}_1$ is an embedding of MeES's; we let $\mathcal{L}(f) = \langle f_e, f_p \rangle$ where $f_e(X) = f(X)$ and $f_p(Y) = f^{-1}(Y)$, for all $X \in \mathcal{L}(\mathcal{E}_0)$ and $Y \in \mathcal{L}(\mathcal{E}_1)$.

Property 4.3.4 *The above definition extends the map $\mathcal{L}$ to a functor from the category* **MeES** *to the category dIDom.*

Proof: The map is well-defined; in fact, suppose first that $X \in \mathcal{L}(\mathcal{E}_0)$. We prove that $f(X) \in \mathcal{L}(\mathcal{E}_1)$: if $A \subseteq_{\mathrm{fin}} f(X)$, then $A = f(B)$ for some $B \subseteq_{\mathrm{fin}} X$, and so $A \in \mathrm{Con}_1$. Moreover, if $e_1, \ldots, e_k$ is a safe sequence in X, then so is $f(e_1), \ldots, f(e_k)$ in $f(X)$. (The proof for f_p is similar). It is clear that $\langle f_e, f_p \rangle$ is an embedding-projection pair. For stability, suppose that $Y \subseteq f_e(X) = f(X)$. Then, take only those elements x of X with $f(x) \in Y$, i.e., consider $X' = X \cap f^{-1}(Y)$. We must prove that X' is a configuration, which is a consequence of Lemma 4.3.8. □

So, $\mathcal{L}$ is a functor; it is also a full functor, as one can prove with much the same techniques as we adopted for the case of (generalized) tolerance spaces (Theorems 4.3.2 and 4.3.4). But, what about faithfulness?

Faithfulness can be disproved in a very easy way. For any two event structures with minimum enabling $\mathcal{E}$ and $\mathcal{E}'$, define nullify$(\mathcal{E}, \mathcal{E}')$ as the structure obtained as follows:

- its events are the (disjoint) union of E and E', plus a special event $*$;
- consistency is given by the union of the two predicates;
- the enabling for events of E is unchanged; every event of E' has the same enablings as before, but with $*$ required to happen, and $*$ itself has no enabling at all.

In practice, nullify$(\mathcal{E}, \mathcal{E}')$ is a structure with the same behaviour (configurations) as $\mathcal{E}$, but with a copy of the structure $\mathcal{E}'$ in it (which does not generate any configuration, though, because of the presence of the event $*$, which acts as a sort of "obstruction"). Clearly $\mathcal{L}(\mathrm{nullify}(\mathcal{E}, \mathcal{E}')) \cong \mathcal{L}(\mathcal{E})$ (and this is really an equality). If $f : \mathcal{E}'_0 \to \mathcal{E}'_1$ is an embedding of MeES's, we can extend this to an embedding $\hat{f} : \mathrm{nullify}(\mathcal{E}, \mathcal{E}'_0) \to \mathrm{nullify}(\mathcal{E}, \mathcal{E}'_1)$ by taking $\hat{f}$ to be the identity on $E \cup \{*\}$. Thus, $\mathcal{L}(\hat{f}) : \mathcal{L}(\mathcal{E}) \to \mathcal{L}(\mathcal{E})$ is the identity always, regardless of how f is chosen. This proves not only that $\mathcal{L}$ is *not faithful*, but also that it is not faithful even for a chosen object in the image of the functor. Thus, to summarize:

Theorem 4.3.9 $\mathcal{L}$: **MeES** $\to$ **dIDom** *is a full functor, and every object of* **dIDom** *is isomorphic to $\mathcal{L}(\mathcal{E})$ for some MeES $\mathcal{E}$. Yet, $\mathcal{L}$ is* not *faithfull, and thus* not *a categorical equivalence.* □

Nonetheless:

Corollary 4.3.4 *$\mathcal{L}(\mathcal{E}_U)$ is a universal object of* **dIDom**.

Proof: For all dI-domains $\mathcal{D}$, consider the following diagram:

$$\begin{tikzcd} & \mathcal{L}(\mathcal{E}_U) & & \mathcal{E}_U \\ \mathcal{D} \arrow[ur, "\mathcal{L}(f_U)\circ\phi"] \arrow[r, "\varphi"] & \mathcal{L}(\mathcal{E}(\mathcal{D})) \arrow[u, "\mathcal{L}(f_U)"'] & & \mathcal{E}(\mathcal{D}) \arrow[u, "f_U"'] \end{tikzcd}$$

where φ is the isomorphism between $\mathcal{D}$ and $\mathcal{L}(\mathcal{E}(\mathcal{D}))$, whose existence is insured by Theorem 4.3.8, and $f_U : \mathcal{E}(\mathcal{D}) \to \mathcal{E}_U$ exists by the universality property of $\mathcal{E}_U$ (Theorem 4.3.5). Then, $\mathcal{L}(f_U) \circ \varphi$ gives the required arrow. $\square$

Homogeneity of $\mathcal{L}(\mathcal{E}_U)$ is not guaranteed, though: it would be interesting to know whether $\mathcal{L}(\mathcal{E}_U)$ is also homogeneous (i.e., if it is isomorphic to the one built in [Dro91]), but this is quite unlikely, because of the way we used to built the homogeneous object in the category $\mathcal{E}_U$.

The problem, here, is that our techniques for the construction of universal homogeneous representations are essentially based on the degree of freedom allowed in building the structure, and this, on the other side, gives structures which are highly non-canonical, in the sense that they actually contain spurious and redundant information. This is precisely the reason for which we have no categorical equivalence; of course, one could think to bypass this problem by imposing "more structure" on the representation side, for example, by giving more constraints in the definition of a MeES, but in that way, even though categorical equivalence is obtained, we have too much structure to be taken into account when building the universal homogeneous object.

4.4 An application to the solution of recursive domain equations

In this section, we shall discuss briefly how one can use the universality results so far obtained in order to solve recursive domain equations. We will not enter into the details of the constructions, and limit ourselves to the very simple case of atomic coherent dI-domains, even though the techniques we explain are suitable to be applied, with slight variations, to all the categories described above.

4.4.1 Domain equations

Before starting, we need introduce the problem, and we do this without any claim of precision or completeness: the interested reader can obtain much more information about domain equations and other possible solution techniques by consulting, e.g., [GS90].

Given a suitable category of domains, one can define many operations (i.e., functors) on the category, each corresponding to a particular semantic construction. For example, given two domains $\mathcal{D}_0$ and $\mathcal{D}_1$, one can consider the new domain $\mathcal{D}_0 \times \mathcal{D}_1$, whose elements are just pairs of elements (from D_0 and D_1) ordered componentwise. This simply corresponds to describing an "object" made up of two components, one of "type" $\mathcal{D}_0$ and the other of "type" $\mathcal{D}_1$, and seems to be the natural semantic counterpart of a typical programming language constructor like **record** in Pascal, or **struct** in C; if the category considered is closed under such products, then $\times$ is a functor on the category, and one can for example wonder whether there is some domain $\mathcal{D}$ such that $\mathcal{D} \cong \mathcal{D} \times \mathcal{D}$.

This is a recursive domain equation, and we would like to find a (minimal) solution for it. The problem of finding solutions to recursive domain equations (expressed in suitable categories) is of primary importance in the field of denotational semantics, where each such equation corresponds to the actual definition of a recursive data-type (e.g., trees, lists or such).

4.4.2 Representing domains as points of the universal domain

In this section we shall explain how one can try to use the universality result for solving recursive domain equations, by representing each domain as a point in the universal domain, and each operation by a (hopefully simple) continuous endofunction of the universal domain.

Before doing this, we prove that the category **CAdIDom** is closed under the operation of stable exponentiation. With this purpose in mind, let us first rephrase Lemma 3.2.1 in the case of atomic coherent dI-domains (induced by tolerance spaces).

Lemma 4.4.1 *Let $f \in [\mathit{Clique}(\mathcal{D}_0) \to_s \mathit{Clique}(\mathcal{D}_1)]$, C be a clique of $\mathcal{D}_0$ and $y \in f(C)$. Then, there is a finite clique $C' \subseteq_{\mathrm{fin}} C$ such that $y \in f(C')$ and, if $C'' \subseteq_{\mathrm{fin}} C$ is such that $y \in f(C'')$, then $C' \subseteq C''$.*

Proof: Since $\{y\} \subseteq f(C)$, use Lemma 3.2.1 and let $C' = M(f, C, \{y\})$. We then have $C' \subseteq C$, $y \in f(C')$. Moreover, if $C'' \subseteq_{\mathrm{fin}} C$ and $y \in f(C'')$, then necessarily $C' \subseteq C''$. □

Now, for any two tolerance spaces $\mathcal{T}_0, \mathcal{T}_1$, let $\mathcal{T} = [\mathcal{T}_0 \to_s \mathcal{T}_1]$ be the tolerance space defined as follows:

- the support set is $X = \{\langle C, y\rangle : C \text{ is a finite clique of } \mathcal{T}_0,\ y \in X_1\}$;
- we define $\langle C, y\rangle \operatorname{co} \langle C', y'\rangle$ if and only if, whenever $C \cup C'$ is a clique, the following hold:
  1. $y \operatorname{co} y'$;
  2. if $y = y'$ then $C = C'$.

What we want to prove is that this new space is exactly the representation of the stable function space. To do this, let us introduce another definition.

If $f \in [\mathrm{Clique}(\mathcal{T}_0) \to_s \mathrm{Clique}(\mathcal{T}_1)]$, define its trace $\mathrm{Tr}(f)$ as the set of all those pairs $\langle C, y\rangle$ where C is a finite clique of $\mathcal{T}_0$, $y \in f(C)$ and, whenever $C' \subseteq C$ with $y \in f(C')$, it happens that $C = C'$.

First, notice that $\mathrm{Tr}(f)$ is always a clique of $[\mathcal{T}_0 \to_s \mathcal{T}_1]$, as explained by:

Lemma 4.4.2 *The set* $\mathrm{Tr}(f)$ *is a clique of* $[\mathcal{T}_0 \to_s \mathcal{T}_1]$.

Proof: Let $\langle C, y\rangle, \langle C', y'\rangle \in \mathrm{Tr}(f)$ and suppose that $C \cup C'$ is a clique. Then, by monotonicity of f, $f(C) \cup f(C') \subseteq f(C \cup C')$. So $y, y' \in f(C \cup C')$ and thus $y \operatorname{co} y'$. Moreover, if $y = y'$, then, by definition of $\mathrm{Tr}(f)$, neither $C \subset C'$ nor $C' \subset C$ hold. By stability, since $C \uparrow C'$, we have $f(C \cap C') = f(C) \cap f(C')$: but $y \in f(C) \cap f(C')$, and so $y \in f(C \cap C')$, which contradicts minimality. $\square$

The map Tr has two important properties, expressed in the following lemmata:

Lemma 4.4.3 *The map* $\mathrm{Tr} : [\mathit{Clique}(\mathcal{T}_0) \to_s \mathit{Clique}(\mathcal{T}_1)] \to \mathit{Clique}([\mathcal{T}_0 \to_s \mathcal{T}_1])$ *is injective.*

Proof: Suppose $\mathrm{Tr}(f) = \mathrm{Tr}(g)$, and let D be a clique of $\mathcal{T}_0$ and $y \in f(D)$. Then, by Lemma 4.4.1, there is a finite clique $C \subseteq D$ such that $y \in f(C)$ and $y \notin f(C')$ for all $C' \subset C$. So $\langle C, y\rangle \in \mathrm{Tr}(f) = \mathrm{Tr}(g)$; but this means that $y \in g(C')$ and thus, by monotonicity of g, also $y \in g(D)$. Thus, finally, $f(D) = g(D)$. $\square$

Lemma 4.4.4 *The map* Tr *is surjective.*

Proof: Let F be a clique of $[\mathcal{T}_0 \to_s \mathcal{T}_1]$: we must build a stable function $f : \mathrm{Clique}(\mathcal{T}_0) \to \mathrm{Clique}(\mathcal{T}_1)$ such that $\mathrm{Tr}(f) = F$. Let, for every clique D of $\mathcal{T}_0$

$$f(D) = \{y \in X_1 : \langle C, y\rangle \in F \text{ for some } C \subseteq_{\mathrm{fin}} D\}.$$

Trivially f is monotone; for continuity, let S be a directed set of cliques; $y \in f(\cup S)$ if and only if $\langle C, y\rangle \in F$ for some $C \subseteq_{\mathrm{fin}} \cup S$, which happens, by directedness, if and only if $C \subseteq_{\mathrm{fin}} D$ for some $D \in S$. This is equivalent to saying that $y \in \cup f(S)$, as required.
For stability, suppose that $D \uparrow D'$, and let $y \in f(D) \cap f(D')$. This means that there are some $C \subseteq_{\mathrm{fin}} D$ and $C' \subseteq_{\mathrm{fin}} D'$ with $\langle C, y\rangle, \langle C', y'\rangle \in F$. But $C \cup C'$ is a clique, and so necessarily $C = C'$. Thus $C \subseteq_{\mathrm{fin}} D \cap D'$ and $\langle C, y\rangle \in F$, which in turn implies $y \in f(D \cap D')$.
Now, suppose that $\langle C, y\rangle \in F$; then clearly $y \in f(C)$: note that we cannot have $y \in f(C')$ for some $C' \subset C$, because otherwise there would be some finite $C'' \subset C$ with $\langle C'', y\rangle, \langle C, y\rangle \in F$ (contradicting the fact that F is a clique). So $\langle C, y\rangle \in \mathrm{Tr}(f)$.
Conversely, if $\langle C, y\rangle \in \mathrm{Tr}(f)$, then $y \in f(C)$, and $y \notin f(C')$ for all $C' \subset C$. This means that there is some $C'' \subseteq C$ with $\langle C'', y\rangle \in F$ but this is not true for any $C'' \subset C$. So $\langle C, y\rangle \in F$, as required. $\square$

As a matter of fact, Tr is actually an (order-)isomorphism:

Theorem 4.4.1 *The function* $\mathrm{Tr} : [\mathit{Clique}(\mathcal{T}_0) \to_s \mathit{Clique}(\mathcal{T}_1)] \to \mathit{Clique}([\mathcal{T}_0 \to_s \mathcal{T}_1])$ *is an isomorphism.*

Proof: The map Tr is well-defined, as proved in Lemma 4.4.2; moreover, it is a bijection (by Lemmata 4.4.3 and 4.4.4).
If $f \sqsubseteq_s g$ then suppose $\langle C, y\rangle \in \mathrm{Tr}(f)$: this means that $y \in f(C)$ and $y \notin f(C')$ for any $C' \subset C$. We know that $f(C) = f(C) \cap g(C)$, and so $y \in g(C)$. Moreover, suppose that C' is such that $C' \subseteq C$ and $y \in g(C')$: then $f(C') = f(C) \cap g(C')$ and so $y \in f(C')$: this implies $C' = C$. So $\langle C, y\rangle \in \mathrm{Tr}(g)$.

Now, for the converse, consider two cliques $C' \subseteq C \in \mathrm{Clique}(\mathcal{D}_0)$. We must prove that $f(C') = f(C) \cap g(C')$. If $y \in f(C')$, by Lemma 4.4.1, there is $C'' \subseteq_{\mathrm{fin}} C'$ such that $y \in f(C'')$ and C'' is minimal with this property. So $\langle C'', y\rangle \in \mathrm{Tr}(f) \subseteq \mathrm{Tr}(g)$; thus $y \in g(C'')$ which implies by monotonicity that $y \in g(C')$. Also $y \in f(C)$ by monotonicity, and so $y \in f(C) \cap g(C')$. Conversely, if $y \in f(C) \cap g(C')$, then by Lemma 4.4.1, there is a minimal $D \subseteq_{\mathrm{fin}} C$ with $y \in f(D)$, and a minimal $D' \subseteq_{\mathrm{fin}} C'$ such that $y \in g(D')$. Then $\langle D, y\rangle \in \mathrm{Tr}(f) \subseteq \mathrm{Tr}(g)$ and $\langle D', y'\rangle \in \mathrm{Tr}(g)$. But $D \cup D' \subseteq C \cup C' \subseteq C$, which is a clique: since $\mathrm{Tr}(g)$ is a clique of $[\mathcal{T}_0 \to_s \mathcal{T}_1]$, we must have $D = D'$. Thus $\langle D', y\rangle \in \mathrm{Tr}(f)$, and so $y \in f(D')$ which implies, by monotonicity, $y \in f(C')$. $\square$

This in particular says that the stable function space $[\mathcal{D}_0 \to_s \mathcal{D}_1]$ is a coherent dI-domain, whenever $\mathcal{D}_0$ and $\mathcal{D}_1$ are such.

At this point, we have paved our way towards the possibility of coding each coherent atomic dI-domain as a point of the universal domain $\mathcal{D}_{\mathrm{R}} = \mathrm{Clique}(\mathcal{T}_{\mathrm{R}})$. In fact, consider any coherent atomic dI-domain $\mathcal{D}$; by the universality property, there will be a stable embedding-projection pair $\langle e, p\rangle : \mathcal{D} \to \mathcal{D}_{\mathrm{R}}$ which in a sense "represents" $\mathcal{D}$ as a subdomain of the universal domain. Now, using Lemma 3.3.1, we know that e and p are both stable, and thus (since the composition of stable functions is also stable) $\pi_{\mathcal{D}} = e \circ p : \mathcal{D}_{\mathrm{R}} \to \mathcal{D}_{\mathrm{R}}$ is a stable function. But then $\pi_{\mathcal{D}} \in [\mathcal{D}_{\mathrm{R}} \to_s \mathcal{D}_{\mathrm{R}}]$, and $[\mathcal{D}_{\mathrm{R}} \to_s \mathcal{D}_{\mathrm{R}}]$ is also a coherent atomic dI-domain, as a consequence of Theorem 4.4.1, and so can be somehow embedded into $\mathcal{D}_{\mathrm{R}}$, by a pair $\langle e_{\to_s}, p_{\to_s}\rangle : [\mathcal{D}_{\mathrm{R}} \to_s \mathcal{D}_{\mathrm{R}}] \to \mathcal{D}_{\mathrm{R}}$. Thus $e_{\to_s}(\pi_{\mathcal{D}})$ is a point of $\mathcal{D}_{\mathrm{R}}$ which faithfully represents the domain $\mathcal{D}$.

At this point, one can try to represent each operation of relevance as an endofunction of $\mathcal{D}_{\mathrm{R}}$, and solve domain equations not *up to isomorphism* in the category **CAdIDom**, but *up to equality* inside the domain $\mathcal{D}_{\mathrm{R}}$. We do not deepen this technique any further, but just observe that this "encoding/decoding" allows one to use standard domain-theoretic results like Knaster-Tarski's Theorem for obtaining solutions to recursive domain equations.

4.4.3 A more direct approach — Solution of $\mathcal{D} \cong \mathcal{D} \oplus \mathbb{I}$

In this last paragraph, we just suggest how one can use a more direct approach to the solution of recursive domain equations, by exploiting the special number-theoretic properties used for building up the universal domain.

In order to do this, we consider a toy example which involves the usage of a special domain operation, called coalesced sum. The *coalesced sum* of two domaind $\mathcal{D}_0$ and $\mathcal{D}_1$ is the domain $\mathcal{D} = \mathcal{D}_0 \oplus \mathcal{D}_1$ defined as follows:

- $D = ((D_0 \setminus \{\bot_0\}) \times \{0\}) \cup ((D_1 \setminus \{\bot_1\}) \times \{1\}) \cup \{\bot\}$;
- $t \sqsubseteq t'$ if and only if ($t = \langle x, 0\rangle$, $t' = \langle x', 0\rangle$ and $x \sqsubseteq_0 x'$) or ($t = \langle y, 1\rangle$, $t' = \langle y', 1\rangle$ and $y \sqsubseteq_1 y'$) or $t = \bot$.

This operation is sketched in Fig. 4.1: in practice, the coalesced sum corresponds simply to the disjoint union of the two domains, modulo the identification of the two bottom elements.

Now, let $\mathbb{I}$ the Sierpinski space, i.e., the (unique) two-point domain $\mathbb{I} = \{\bot, \top\}$. How could we solve the recursive domain equation $\mathcal{D} \cong \mathcal{D} \oplus \mathbb{I}$ in the category **CAdIDom**?

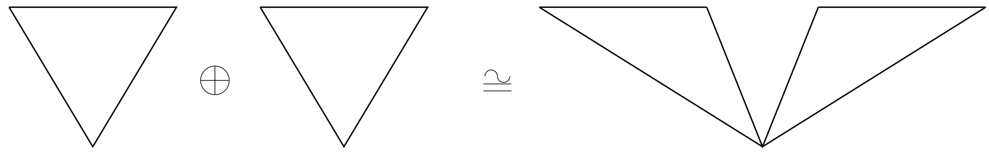


Figure 4.1: The coalesced sum of two domains.

One way to do this is the following: first, observe that every coherent atomic dI-domain $\mathcal{D}$ can be somehow represented as the domain of cliques of some tolerance space (i.e., $\mathcal{D} \cong \mathrm{Clique}(\mathcal{T})$), and the latter can be embedded in the universal Rado's tolerance space. Thus, we can associate to each domain a set of positive integers, corresponding to the points of Rado's tolerance space to which the elements of $\mathcal{T}$ are mapped by the embedding. Now, if we find an operation among sets of integers which mimicks the construction of interest (in our case: the coalesced sum), we can use it to solve the domain equation.

To see how this can work, consider the function $F : \wp(\omega) \times \wp_{\mathrm{fin}}(\omega) \to \wp(\omega)$ defined as follows: if $0 < x_1 < x_2 < \ldots < x_n$, and $0 \notin A$, we define

$$F(A, \{x_1, \ldots, x_n\}) = A \cup \{x'_1, \ldots, x'_n\},$$

where, for all $k = 1, \ldots, n$, letting $I_k = \{i = 1, \ldots, k-1 :$ the x_i-th bit of x_k is a $1\}$, we define

$$x'_k = (x_k - \sum_{i \in I_k} 2^{x_i}) 2^{\max A} + \sum_{i \in I_k} 2^{x'_i}.$$

The point here is that this function represents the coalesced sum of two coherent atomic dI-domains, in the precise sense we are now going to explain. First, we need a definition and two lemmata. For two given tolerance spaces $\mathcal{T}_0$, $\mathcal{T}_1$, we define their disjoint union $\mathcal{T}_0 + \mathcal{T}_1$ as that tolerance space whose underlying set is just the disjoint union of the sets X_0 and X_1, and where tolerance is the disjoint union of the two tolerance relations.

First note that

Lemma 4.4.5 *Let $\mathcal{T}_0, \mathcal{T}_1$ be two tolerance spaces. Then Clique$(\mathcal{T}_0 + \mathcal{T}_1) \cong$ Clique$(\mathcal{T}_0) \oplus$ Clique$(\mathcal{T}_1)$.*

Proof: A clique of $\mathcal{T}_0 + \mathcal{T}_1$ cannot contain elements coming from the two different components (because they are not in the tolerance relation). Thus, either it is a clique of $\mathcal{T}_0$ or a clique of $\mathcal{T}_1$, unless it is the empty clique. $\square$

As a matter of fact:

Lemma 4.4.6 *Let $0 \notin A \subseteq \omega$ and $B \subseteq_{\mathrm{fin}} \omega \setminus \{0\}$. Then $\mathcal{T}_{\mathrm{R}}^{F(A,B)} \cong \mathcal{T}_{\mathrm{R}}^{A} + \mathcal{T}_{\mathrm{R}}^{B}$.*

Proof: Let $B = \{x_1 < x_2 < \ldots < x_k\}$: we prove the statement by induction on k. If $k = 0$, then $F(A, B) = A$ and we have nothing to prove. Suppose now that $B = \{x_1 < \ldots < x_k < x_{k+1}\}$ and let $C = B \setminus \{x_{k+1}\}$. By induction hypothesis, $\mathcal{T}_{\mathrm{R}}^{F(A,C)} \cong \mathcal{T}_{\mathrm{R}}^{A} + \mathcal{T}_{\mathrm{R}}^{C}$. Now

$$F(A, B) = F(A, C) \cup \{x = (x_{k+1} - \sum_{i \in I_{k+1}} 2^{x_i}) 2^{\max A} + \sum_{i \in I_{k+1}} x'_i\}.$$

The element x is not connected to any element of the A-component, since $2^{\max A}$ is strictly greater than any element of A, and so is any x'_i; moreover, x is connected to x'_i in the C-component if and only if $i \in I_{k+1}$, i.e., iff x_{k+1} was connected to x_i. □

So, to summarize:

Theorem 4.4.2 *Let $0 \notin A, B \subseteq \omega$, with B finite. Then:*

$$Clique(\mathcal{T}_{\mathrm{R}}^{F(A,B)}) \cong Clique(\mathcal{T}_{\mathrm{R}}^{A}) \oplus Clique(\mathcal{T}_{\mathrm{R}}^{B}).$$

Proof: A consequence of Lemmata 4.4.5 and 4.4.6. □

Note that, in the very special case when B is a singleton, the function F can be defined by

$$F(A, \{x\}) = A \cup \{x \cdot 2^{\max A}\}.$$

Now, we can solve our domain equation using the following easy observation: the Sierpinski space can be represented by $\mathbb{I} \cong \mathrm{Clique}(\mathcal{T}_{\mathrm{R}}^{\{1\}})$ (because a one-point tolerance space has exactly two cliques). Thus, looking for the solution of the domain equation $\mathcal{D} \cong \mathcal{D} \oplus \mathbb{I}$, is equivalent to finding a subset A of ω such that $\mathrm{Clique}(\mathcal{T}_{\mathrm{R}}^{A}) \cong \mathrm{Clique}(\mathcal{T}_{\mathrm{R}}^{A}) \oplus \mathbb{I}$ which, by the previous theorem, is isomorphic to $\mathcal{T}_{\mathrm{R}}^{F(A,\{1\})}$.

In other words, we have reduced our problem to solving the recursive equation $A = F(A, \{1\})$ on the domain $\wp(\omega)$. Now, using the fixed-point theorem, we obtain the following increasing chain:

$$\begin{array}{lcl}
A_0 & = & \emptyset \\
A_1 & = & F(A_0, \{1\}) = \{1\} \\
A_2 & = & F(A_1, \{1\}) = \{1, 2^2\} \\
A_3 & = & F(A_2, \{1\}) = \{1, 2^2, 2^4\} \\
\ldots & & \\
A_k & = & \{2^i : i < k\}.
\end{array}$$

Thus, A is the set of all powers of two, and $\mathcal{T}_{\mathrm{R}}^{A}$ is clearly the (unique, up to isomorphism) countable totally disconnected tolerance space. Finally, the solution $\mathcal{D} = \mathrm{Clique}(\mathcal{T}_{\mathrm{R}}^{A})$ is simply the flat domain $\omega_\perp$ of natural numbers shown in Fig. 4.2 (because the cliques of the countable totally disconnected tolerance space are just the empty clique and all the singleton sets).

4.5 An alternative universal construction

In this section, we shall provide an alternative universal construction for the category of coherent dI-domains, based on the concepts of trace automata and trace language. This construction was presented in [BCS93], and requires some technical background which we shall introduce next.

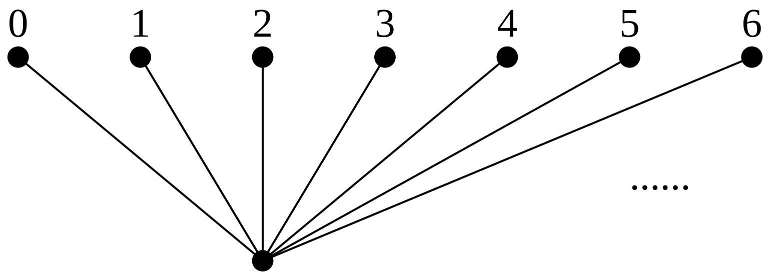


Figure 4.2: The flat domain $\omega_\perp$ of natural numbers, which is the solution of $\mathcal{D} \cong \mathcal{D} \oplus \mathbb{I}$

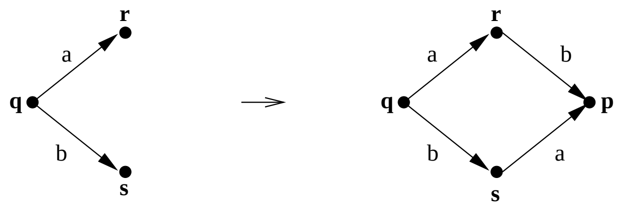


Figure 4.3: The commutativity condition

4.5.1 Trace automata and computation sequences

The notion of concurrent automaton we shall use is a particular instance of the *trace automata* in the sense of Stark [Sta89b], although we do not require the existence of ϵ-transitions. Our machines, the *full* trace automata, are slight variations of the *asynchronous transition systems* of Shields [Shi85] and Bednarczyk [Bed88]. Besides the intrinsic interest of this model in the field of concurrency theory, we shall observe later that there is an exact correspondence between the set of computations of such an automaton and the set of configurations of a prime event structure, which can be extended to an isomorphism of partial orders when prefix ordering is defined on computations. In order to maintain this correspondence it is necessary to admit countable alphabets and sets of states for automata. Furthermore, we shall interpret symbols of the alphabets as "events", instead of "actions", because this confusion is harmless in the present context.

A *trace automaton* is a structure $\mathcal{A} = (E, \|_E, Q, T, *)$, where E is a countable set of *events*, $\|_E$ is an irreflexive and symmetric relation over E (the *concurrency relation*), Q is a set of *states* with $* \in Q$ (the *initial state*), and $T \subseteq (Q \times E \times Q)$ is a set of *transitions.* A transition of the form $t = (q, a, r)$ will also be written as $t : q \xrightarrow{a} r$, and for such a transition we let $\mathbf{dom}(t) = q$, $\mathbf{cod}(t) = r$ and $\mathbf{event}(t) = a$. These data are required to satisfy the following conditions:

Disambiguation If $q \xrightarrow{a} r, q \xrightarrow{a} r'$ then $r = r'$;

Commutativity If $q \xrightarrow{a} r, q \xrightarrow{b} s$ and $a \|_E b$, then $\exists p \in Q.\, r \xrightarrow{b} p, s \xrightarrow{a} p$, as in the diagram of Fig. 4.3.

A trace automaton is *full* if it satisfies:

Fullness if $q \xrightarrow{a} r, r \xrightarrow{b} p$ and $a \|_E b$ then $q \xrightarrow{b} s$ for some state $s \in Q$, as in the diagram of Fig. 4.4.

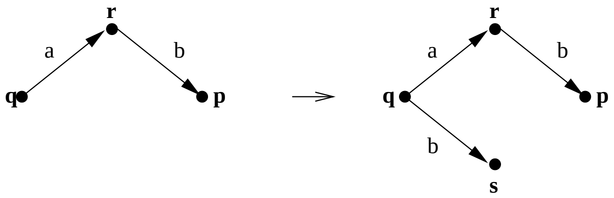


Figure 4.4: The fullness condition

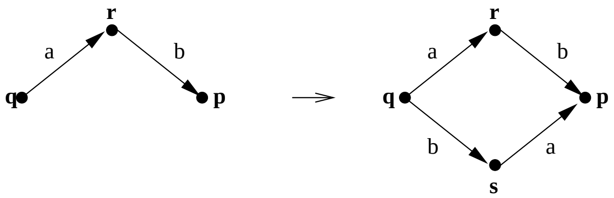


Figure 4.5: An equivalent version of the fullness condition

Observe that the state s whose existence is guaranteed by the fullness condition is uniquely determined by disambiguation and, by commutativity, we can equivalently represent the condition as in the diagram of Fig. 4.5.
The disambiguation condition corresponds to an assumption of determinism, even if the automata are not completely specified. Also, there is no notion of final state, so acceptance in this context is determined by maximality of computation sequences. Intuitively, fullness amounts to requiring that computation sequences of a trace automaton represent all possible interleaving of events in a run of a concurrent process.

A (finite) *computation sequence* of an automaton[6] $\mathcal{A} = (E, \|_E, Q, T, *)$ is either empty or else a sequence $\gamma = t_1 \ldots t_n$ of transitions of $\mathcal{A}$ such that $\mathbf{dom}(t_{i+1}) = \mathbf{cod}(t_i)$ for $i = 1, \ldots, n-1$. We define $\mathbf{dom}(\gamma) = \mathbf{dom}(t_1)$ and $\mathbf{cod}(\gamma) = \mathbf{cod}(t_n)$, and the length of this computation sequence is $|\gamma| = n$. The set of finite computation sequences of $\mathcal{A}$ having $*$ as domain is denoted by $\mathbf{CS}^\circ(\mathcal{A})$. We also have infinite computation sequences of $\mathcal{A}$, of the form $\gamma = t_1 t_2 t_3 \ldots t_i \ldots$, and the set of finite and infinite computation sequences of $\mathcal{A}$ whose domain is the initial state will be denoted by $\mathbf{CS}(\mathcal{A})$. Two computation sequences γ, δ are composable when $\mathbf{cod}(\gamma) = \mathbf{dom}(\delta)$, and their composition is the computation sequence $\gamma\delta$. We say that γ is a prefix of δ, and write $\gamma \leq \delta$, when there is a computation sequence ξ such that $\gamma\xi = \delta$.

Observe that the disambiguation condition determines a bijective correspondence between computation sequences and the finite or infinite words obtained by concatenating the symbols labelling their transitions. We shall exploit this fact and often represent computation sequences by the corresponding words.

As a better description of the partial order of events in a distributed environment, it is possible to identify sequences up to permutations of adjacent symbols whenever they represent concurrent actions. Thus we are naturally led to consider traces as descriptions of concurrent computations, and indeed full trace automata act as acceptors for a special

[6]Throughout this section, the word "automaton" is always used to mean "trace automaton".

class of trace languages (in the sense of Mazurkiewicz [Maz77]; see also [DR95] for a survey on the subject).

A *concurrent alphabet* is a pair $\mathfrak{A} = (A, \|_A)$, where A is a countable set and $\|_A$ is an irreflexive and symmetric relation over A. The *trace monoid* $\Theta(\mathfrak{A})$ is the quotient of A^* modulo the smallest congruence $\sim_A$ containing all pairs (ab, ba) for $a\|_A b$. An element of $\Theta(\mathfrak{A})$ is called a *trace*. A subset of $\Theta(\mathfrak{A})$ is called a *trace language*.

By $\mathrm{Alph}([w])$, for a trace $[w]$ over $\mathfrak{A}$, we denote the set of symbols occurring in any of its representatives $w \in A^*$. The set of traces can be partially ordered by the prefix relation: for $s, t \in \Theta(\mathfrak{A})$:

$$s \leq_{\mathfrak{A}} t \text{ iff } \exists u \in \Theta(\mathfrak{A}).\ su = t.$$

Obviously $[\epsilon]$, where ϵ denotes the empty word, is the least element of the partially ordered set $\Theta(\mathfrak{A})$.

We associate in a natural way to each computation sequence $\gamma = t_1 \ldots t_n$ the trace $\mathrm{tr}(\gamma) = [\mathbf{event}(t_1) \ldots \mathbf{event}(t_n)]$. This mapping induces an equivalence relation on computation sequences, which coincides with the "permutation equivalence" defined by Stark [Sta89b], as well as a notion of prefix.

Given $\gamma, \delta \in \mathbf{CS}^\circ(\mathcal{A})$, let $\gamma \sim \delta$ whenever $\mathrm{tr}(\gamma) = \mathrm{tr}(\delta)$, and in this case say that two sequences γ, δ are *equivalent up to permutations*. The relation $\precsim$ over $\mathbf{CS}^\circ(\mathcal{A})$ (*prefix up to permutations*) is the transitive closure of $\leq \cup \sim$, and the partially ordered set $D^\circ(\mathcal{A})$ is defined to be $\mathbf{CS}^\circ(\mathcal{A})/\sim$, ordered by $\precsim$.

Observe that, if $\mathcal{A}$ is a full automaton, the function

$$\mathrm{word} : \mathbf{CS}^\circ(\mathcal{A}) \to \bigcup\{\mathrm{tr}(\gamma), \gamma \in \mathbf{CS}^\circ(\mathcal{A})\},$$

defined by the mapping

$$\gamma = t_1 \ldots t_n \longmapsto \mathbf{event}(t_1) \ldots \mathbf{event}(t_n)$$

is bijective. In other words, every representative of the trace of a finite computation sequence γ is the word associated to some computation sequence $\delta \in \mathbf{CS}^\circ(\mathcal{A})$, where $\gamma \sim \delta$.

4.5.2 Prime event structures and coherent dI-domains

Coherent dI-domains, introduced in Chapter 3, also arise as domains of configurations of prime event structures with a binary conflict relation (see [Win91]).

A *prime event structure* [Win80] is a triple $\mathcal{E} = (E, \leq, \#)$, where E is a countable set (of "events"), $\leq$ is a partial ordering on the events (the "causality relation") and $\#$ is a binary irreflexive and symmetric relation on the events (the "conflict" relation) satisfying:

- for all $e \in E$, the set $\downarrow e = \{e', e' \leq e\}$ is finite;
- if $e\#e'$ and $e' \leq e''$, then $e\#e''$.

Note that this is a special instance of the definition of event structure where conflict is specified by a binary relation, and enabling is given by a partial order.

If $\mathcal{E} = (E, \leq, \#)$ is a prime event structure, we can associate to it the poset $\mathcal{L}(\mathcal{E})$ consisting of its *configurations* (ordered by inclusion). These are the subsets x of E such that

- for $e, e' \in x$, $\neg(e\#e')$;
- if $e \in x$ and $e' \leq e$ then $e' \in x$.

More precisely, we have the following representation theorem which combines results of Nielsen, Plotkin and Winskel [NPW81] and Winskel [Win87] (see also [BB95] for another representation theorem in terms of special causally labelled trees):

Theorem 4.5.1 *For every prime event structure $\mathcal{E}$, the poset $\mathcal{L}(\mathcal{E})$ is a coherent dI-domain. Moreover, every coherent dI-domain $\mathcal{D}$ is isomorphic to the domain of configurations of a prime event structure $\mathcal{E}(\mathcal{D}) = (E_D, \leq, \#)$.* □

Observe that an event $e \in E$ of a prime event structure $\mathcal{E}$ can be mapped to the configuration $\downarrow e = \{e' \in E, e' \leq e\}$, and that configurations of this form can be characterized order-theoretically as the complete primes of $\mathcal{L}(\mathcal{E})$.

For an arbitrary coherent dI-domain $\mathcal{D}$ we define the prime event structure $\mathcal{E}(\mathcal{D}) = (E_D, \leq, \#)$ by taking E_D as the set $\mathrm{Pr}(D)$ of complete primes of $\mathcal{D}$, $\leq$ as the restriction of the ordering on $\mathcal{D}$ to $\mathrm{Pr}(D)$ and setting $e\#e'$ if and only if $\neg(e \uparrow e')$. Then $\mathcal{E}(\mathcal{D})$ satisfies $\mathcal{L}(\mathcal{E}(\mathcal{D}))) \cong \mathcal{D}$.

4.5.3 Full trace automata and coherent dI-domains

There is a well known representation theory that relates categories of domains to other structures of a "syntactical" nature; some examples have been presented in the first sections of this Chapter. More examples of this theory are the equivalence of domains with *information systems* (Scott [Sco82]), of concrete domains with the *information matrices* of Kahn and Plotkin [KP78] (also called *concrete data structures* in Curien [Cur86]), and the representation already mentioned of coherent dI-domains as families of configurations of a prime event structure.

In this section we examine the connections between classes of distributive domains and concurrent automata, providing alternative representation results for these domains. This research direction was initiated by Shields [Shi82], Bednarczyk [Bed88], Stark [Sta89a] (whose trace automata exactly correspond to the more general kind of event structures presented in [Win87]), and has been further developed in [Dro90].

The two main examples of dI-domains for the purpose of the present section are the domain of all the finite and infinite traces over a concurrent alphabet $\mathfrak{A}$ ordered by prefix, and the domain of finite and infinite computations of a full trace automaton. It turns out that every coherent dI-domain can be represented as the domain of computations of an automaton of this kind, providing an alternative representation for this class of domains. For a different proof of this result see [Win91], where it is attributed to Bednarczyk [Bed88]. The systematic use of properties of embedding-projection pairs in our proofs will turn out to be useful in proving the universality of the coherent dI-domain constructed in the next subsection.

Let $\mathcal{E} = (E, \leq, \#)$ be a prime event structure. The finite configurations of $\mathcal{E}$ form a set $\mathcal{L}^\circ(\mathcal{E})$ whose properties will be used in the construction of a full trace automaton $\mathcal{A}(\mathcal{D})$

from (the event structure $\mathcal{E}(\mathcal{D})$ associated with) a coherent dI-domain $\mathcal{D}$. Indeed, given any prime event structure $\mathcal{E}$, define an automaton

$$\mathcal{A}(\mathcal{E}) = (E, \|_E, Q_E, T_E, *_E)$$

having as events just the events of $\mathcal{E}$ with the concurrency relation determined by the clause

$$e\|_E e' \text{ if and only if } \neg(e \leq e') \ \wedge\ \neg(e' \leq e) \ \wedge\ \neg(e\#e').$$

The states of $\mathcal{A}(\mathcal{E})$ are the elements of $\mathcal{L}^\circ(\mathcal{E})$, with $*_E = \emptyset$, and $x \xrightarrow{e} y$ if and only if $y = x \cup \{e\}$ and $e \notin x$.

In the sequel, we use $\lessdot$ to denote the covering relation associated to a partial order. A *covering chain* from x to y is a chain $x = x_0 \lessdot x_1 \lessdot x_2 \lessdot \ldots \lessdot x_n = y$.

Lemma 4.5.1 *Given $x, y \in \mathcal{L}^\circ(\mathcal{E})$ with $x \subseteq y$, there exists a covering chain from x to y.*

Proof: First observe that, for $x, y \in \mathcal{L}^\circ(\mathcal{E})$, $x \lessdot y$ if and only if $y = x \cup \{e\}$ for some $e \notin x$. The proof is by induction on the cardinality of $y \setminus x$. The basis is obvious, so let $y = x \cup \{e_1, \ldots, e_n\}$. There exists an $e_i \in \{e_1, \ldots, e_n\}$ which is minimal with respect to the ordering on E, so $x \cup \{e_i\} \in \mathcal{L}^\circ(\mathcal{E})$. Then $|y \setminus (x \cup \{e_i\})| < |y \setminus x|$, and the conclusion follows by an application of the induction hypothesis to $x \cup \{e_i\} \subseteq y$. $\square$

The following Lemma states the Jordan-Dedekind property for the partially ordered set of finite configurations of a prime event structure $\mathcal{E}$. Its proof follows from the observation that the length of any covering chain from x to y is completely determined by the cardinality of the difference $y \setminus x$.

Lemma 4.5.2 *In the partial order $\mathcal{L}(\mathcal{E})$ all covering chains between any two elements have the same length.* $\square$

Given $x \in \mathcal{L}^\circ(\mathcal{E})$ and a covering chain C of the form

$$x = x_0 \lessdot x_1 \lessdot x_2 \lessdot \ldots \lessdot x_{n-1} \lessdot x_n = y$$

from x to y, define $\text{word}(C) \in E^*$ as the sequence of events $e_1 \ldots e_n$, where $x_i = x_{i-1} \cup \{e_i\}$ for $i = 1, \ldots, n$. Observe that the pair $(E, \|_E)$ is a concurrent alphabet, so the concurrency relation $\|_E$ induces an equivalence relation $\sim_E \subseteq E^* \times E^*$.

Lemma 4.5.3 *For $x, y \in \mathcal{L}^\circ(\mathcal{E})$ and any two covering chains $C = x \lessdot x_1 \lessdot x_2 \lessdot \ldots \lessdot x_{n-1} \lessdot x_n = y$, $C' = x \lessdot x'_1 \lessdot x'_2 \lessdot \ldots \lessdot x'_{n-1} \lessdot x'_n = y$ from x to y,*

$$\text{word}(C) \sim_E \text{word}(C').$$

Proof: By Lemma 4.5.2, C and C' have the same length n, and the proof is by induction on n. Assume that the chains are not empty (otherwise the property is trivially satisfied), and let $i+1$ be the first index at which the two chains differ. Then $x_{i+1} = x_i \cup \{e_{i+1}\}$ and $x'_{i+1} = x_i \cup \{e'_{i+1}\}$ where $e_{i+1}\|_E e'_{i+1}$. So $x_i \cup \{e_{i+1}, e'_{i+1}\}$ is again a configuration and, by

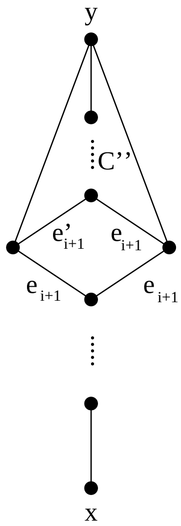


Figure 4.6: Illustration of the proof of Lemma 4.5.3

Lemma 4.5.1, there exists a chain C'' from $x_i \cup \{e_{i+1}, e'_{i+1}\}$ to y. By induction hypothesis we have that

$$e'_{i+1}\mathrm{word}(C'') \sim_E \mathrm{word}(x_{i+1} \lessdot \ldots \lessdot x_n)$$

and

$$e_{i+1}\mathrm{word}(C'') \sim_E \mathrm{word}(x'_{i+1} \lessdot \ldots \lessdot x'_n).$$

Now $e_{i+1}e'_{i+1}\mathrm{word}(C'') \sim_E e'_{i+1}e_{i+1}\mathrm{word}(C'')$, so

$$\mathrm{word}(x_i \lessdot x_{i+1} \lessdot x_{i+2} \lessdot \ldots \lessdot x_{n-1} \lessdot x_n = y)$$

is $\sim_E$-equivalent to

$$\mathrm{word}(x_i \lessdot x'_{i+1} \lessdot x'_{i+2} \lessdot \ldots \lessdot x'_{n-1} \lessdot x'_n = y)$$

and finally also the two words

$$\mathrm{word}(x = x_1 \lessdot x_2 \lessdot x_3 \lessdot \ldots \lessdot x_i \lessdot x_{i+1} \lessdot \ldots \lessdot x_{n-1} \lessdot x_n = y)$$

and

$$\mathrm{word}(x = x_1 \lessdot x_2 \lessdot x_3 \lessdot \ldots \lessdot x_i \lessdot x'_{i+1} \lessdot \ldots \lessdot x'_{n-1} \lessdot x'_n = y)$$

are $\sim_E$-equivalent. $\square$

The above proof is illustrated in Fig. 4.6.

Proposition 4.5.1 *The automaton $\mathcal{A}(\mathcal{E})$ is full.*

Proof: The disambiguation condition on $\mathcal{A}(\mathcal{E})$ is immediate. Assume now that $x \xrightarrow{e} y$ and $x \xrightarrow{e'} y'$ and that $e \|_E e'$. Then $z = x \cup \{e, e'\} \in Q_E$ because this set is again a finite configuration of $\mathcal{E}$, so we have $y \xrightarrow{e'} z$ and $y' \xrightarrow{e} z$. Finally, let $x \xrightarrow{e} y$ and $y \xrightarrow{e'} z$, where

$e\|_E e'$. This entails that $x \cup \{e'\}$ is again a finite configuration of $\mathcal{E}$, in fact it is consistent being a subset of a configuration, and downward closed as e and e' are incomparable, so $x \xrightarrow{e'} x \cup \{e'\}$. □

We can now define a function $\Re$ which maps a $\sim_E$-equivalence class of a computation sequence γ of $\mathcal{A}(\mathcal{E})$ into the finite configuration $\mathbf{cod}(\gamma)$ of $\mathcal{E}$.

Theorem 4.5.2 *The mapping*

$$\Re : D^\circ(\mathcal{A}(\mathcal{E})) \to \mathcal{L}^\circ(\mathcal{E})$$

is an order isomorphism.

Proof: If γ is a finite computation sequence of $\mathcal{A}(\mathcal{E})$, then its $\sim_E$-equivalence class is exactly the set of covering chains from $\emptyset$ to $\mathbf{cod}(\gamma)$: this entails surjectivity by Lemma 4.5.1 and also, by Lemma 4.5.3, that any two equivalence classes of computation sequences having the same codomain are equal, showing the injectivity of $\Re$. Finally, it is straightforward to check that $\Re$ and its inverse are both monotone. □

In the sequel we shall denote by $\overline{\mathfrak{A}}$ the domain $\mathrm{Idl}(\Theta(\mathfrak{A}), \leq)$.

Lemma 4.5.4 *For every full trace automaton $\mathcal{A}$ over the concurrent alphabet $\mathfrak{E}$, there exists a stable embedding of the domain $D(\mathcal{A})$ into $\overline{\mathfrak{E}}$, where $D(\mathcal{A})$ is the ideal completion of $D^\circ(\mathcal{A})$.*

Proof: The function $\mathrm{tr} : \mathbf{CS}^\circ(\mathcal{A}) \to \Theta(\mathfrak{E})$ can be lifted to a function $D^\circ(\mathcal{A}) \to \Theta(\mathfrak{E})$. Stark has proved in [Sta89b], Theorem 4, that this function, extended by continuity to $D(\mathcal{A})$ and $\overline{\mathfrak{E}}$, is an additive injection that satisfies the first condition of Lemma 3.3.2. It also satisfies the second condition as a consequence of the fullness of $\mathcal{A}$, so the result follows.□

The following result is well known: proofs can be found, for example, in [RT91] and [Win91].

Theorem 4.5.3 *For any concurrent alphabet $\mathfrak{A}$, $\overline{\mathfrak{A}}$ is a coherent dI-domain.* □

We can finally state our representation result for coherent dI-domains:

Theorem 4.5.4 1. *If $\mathcal{A}$ is a full trace automaton, then $D(\mathcal{A})$ is a coherent dI-domain.*

2. *For every coherent dI-domain $\mathcal{D}$ there exists a concurrent alphabet $\mathfrak{E} = (E_D, \|_{E(D)})$ and a full trace automaton $\mathcal{A}(\mathcal{D}) = (E_D, \|_{E(D)}, Q_{E(D)}, T_{E(D)}, *_{E(D)})$ such that the domain $D(\mathcal{A}(\mathcal{D}))$ is isomorphic to $\mathcal{D}$.*

Proof: Part (1). It follows from Theorem 4.5.3 using Lemma 4.5.4 and Proposition 3.3.2. *Part (2).* Take $\mathcal{A}(\mathcal{D})$ as the full automaton $\mathcal{A}(\mathcal{E}(\mathcal{D}))$. Then, using Theorems 4.5.2 and 4.5.1 we have that $D(\mathcal{A}(\mathcal{E}(\mathcal{D}))) \cong \mathcal{L}(\mathcal{E}(D)) \cong \mathcal{D}$. □

4.5.4 Construction of a universal coherent dI-domain

The representation of coherent dI-domains in terms of computation sequences of suitable trace automata can be used to build a universal coherent dI-domain, using once more the universal Rado's tolerance space $\mathcal{T}_{\mathrm{R}}$, this time interpreted as a concurrent alphabet.

The following classical result of trace theory is useful for studying the poset $\Theta(\mathfrak{A})$ of traces over a concurrent alphabet $\mathfrak{A}$:

Lemma 4.5.5 (Levi's lemma for traces; see [DR95]) *If $\mathfrak{A} = (A, \|_A)$ is a concurrent alphabet and $ss' = tt'$ for $s, s', t, t' \in \Theta(\mathfrak{A})$, then there are $u_1, u_2, u_3, u_4 \in \Theta(\mathfrak{A})$ such that $s = u_1u_2, t = u_1u_3, s' = u_3u_4, t' = u_2u_4$, and $\mathrm{Alph}(u_2) \times \mathrm{Alph}(u_3) \subseteq \|_A$.* □

The following proposition gives the exact relation between embeddings of concurrent alphabets (i.e., tolerance spaces) and strong ideals of the associated trace domain.

Proposition 4.5.2 *Assume that $f : \mathfrak{A} \to \mathfrak{B}$ is an embedding of the concurrent alphabets (seen as tolerance spaces) $\mathfrak{A} = (A, \|_A)$, $\mathfrak{B} = (B, \|_B)$. Then f, extended to traces, gives an order-isomorphism between $\Theta(\mathfrak{A})$ and a subset M of $\Theta(\mathfrak{B})$, and:*

1. *If $s \leq_{\mathfrak{B}} t$ and $t \in M$, then also $s \in M$;*
2. *If $s, t \in M$ and $s \uparrow t$, then $s \sqcup t$ exists and is an element of M.*

Proof: If f is extended in the canonical way to a function on $A^* \to B^*$, then, by the very definition of embedding of tolerance spaces, one has that, for all $v, w \in A^*$, $v \sim_A w$ if and only if $f(v) \sim_A f(w)$. So, f induce a bijection between $\Theta(\mathfrak{A})$ and a subset of M of $\Theta(\mathfrak{B})$, which respects the prefix ordering. In the rest of the proof, we shall get rid of the isomorphism and simply assume that $A \subseteq B$ and $\|_A = \|_B \cap A^2$ (so $M = \Theta(\mathfrak{A}) \subseteq \Theta(\mathfrak{B})$.

For the first part of the statement, if $s = [u]_{\mathfrak{B}}$ and $t = [uv]_{\mathfrak{A}}$, then $u \in A^*$ and $s = [u]_{\mathfrak{A}}$, so $s \in \Theta(\mathfrak{A})$. For the second part, if $s \uparrow t$, then there exist s', t' such that $ss' = tt'$. By Lemma 4.5.5 there are u_1, u_2, u_3, u_4 such that $s = u_1u_2$, $s' = u_3u_4$, $t = u_1u_3$ and $t' = u_2u_4$ with $\mathrm{Alph}(u_2) \times \mathrm{Alph}(u_3) \subseteq \|_A$. Hence $u = u_1u_2u_3 = u_1u_3u_2 \in \Theta(\mathfrak{A})$. Clearly $s, t \leq_{\mathfrak{A}} u$; assume that for some s'', t'' we have $ss'' = tt''$: then $u_1u_2s'' = u_1u_3t''$ and so, by the cancellation property of the free partially commutative monoid $\Theta(\mathfrak{A})$ (see, for example, Diekert [Die90]), $u_2s'' = u_3t''$. By another application of Lemma 4.5.5 we obtain a decomposition $u_2 = \alpha\beta$, $s'' = \gamma\delta$, $u_3 = \alpha\gamma$ and $t'' = \beta\delta$, with each symbol occurring in β concurrent with all symbols in γ. This entails that $\alpha = [\epsilon]$, so $u_2 = \beta$, $u_3 = \gamma$, and $s'' = u_3\delta$, $t'' = u_2\delta$. Finally we conclude that $u \leq_{\mathfrak{A}} ss''$, so $u = s \sqcup t$. □

We thus obtain, as a consequence of Proposition 4.5.2, the following Corollary:

Corollary 4.5.1 *If $\mathfrak{A}$, $\mathfrak{B}$ are concurrent alphabets and there exists an embedding of $\mathfrak{A}$ into $\mathfrak{B}$, then there is a stable embedding-projection pair from $\overline{\mathfrak{A}}$ to $\overline{\mathfrak{B}}$.*

Proof: In the light of Proposition 3.3.2, using Proposition 4.5.2, we obtain that $\mathrm{Idl}(\Theta(\mathfrak{A}))$ is (isomorphic to) a strong ideal of $\mathrm{Idl}(\Theta(\mathfrak{B}))$. By definition of $\overline{\mathfrak{A}}$ and $\overline{\mathfrak{B}}$, using the second part of Proposition 3.3.1, we obtain the result. □

We can finally state the main result of this section:

Theorem 4.5.5 *The domain $\overline{\mathcal{T}_\mathrm{R}}$ is a universal domain in the category of (countable) coherent dI-domains with stable embedding-projection pairs as morphisms.*

Proof: Given any countable coherent dI-domain $\mathcal{D}$ use Theorem 4.5.4 in order to build the full automaton $\mathcal{A}(\mathcal{D})$ over the concurrent alphabet $\mathfrak{E} = (E_D, \|_{E(D)})$. Its domain of computations $D(\mathcal{A}(\mathcal{D})) \cong \mathcal{D}$ can be stably embedded, by Lemma 4.5.4, into the domain $\overline{\mathfrak{E}}$, which in turn can be rigidly embedded into $\overline{\mathcal{T}_\mathrm{R}}$ by Theorem 4.3.1, Corollary 4.5.1. □

An important question whose answer is not known at present is whether the domain $\overline{\mathcal{T}_\mathrm{R}}$ is isomorphic to the universal coherent dI-domain whose existence is proved in [Dro91]. In particular, it would be interesting to know whether $\overline{\mathcal{T}_\mathrm{R}}$ is homogeneous, because in this case the isomorphism could be established as a consequence of the uniqueness of the universal homogeneous domain of Theorem 4.2.1.

4.6 A note on probabilistic constructions

In this section, we shall discuss in some detail a probabilistic technique which can be used as an alternative way to show the existence of universal homogeneous representations; in particular, our starting point will be the result of Erdős and Rényi [ER63] (discussed at length by Cameron [Cam90]; see also [ES74]), which essentially present a probabilistic proof of existence for the universal Rado's graph. We will show how this could be extended to generalized tolerance spaces and MeES's. In this section, we shall be a bit vague in our notation, and most of the proofs will be only sketched, because a pedantic discussion of probabilistic matters is beyond the scope of this thesis.

4.6.1 Probabilistic construction of Rado's graph

We first present the Erdős-Rényi universality proof for the category **TolSp**. Let $\mathcal{G}$ be the set of undirected graphs[7] with node set ω. Let $\mathbb{A}$ be the smallest family of subsets of $\mathcal{G}$ satisfying the following constraints:

1. for every edge e, the set $\mathcal{G}_e = \{G : e \text{ is an edge of } G\}$ is an element of $\mathbb{A}$;
2. $\mathcal{G}$ is itself an element of $\mathbb{A}$;
3. if $S \in \mathbb{A}$, then also $\mathcal{G} \setminus S \in \mathbb{A}$;
4. if $\mathbb{S}$ is a countable subset of $\mathbb{A}$, then $\cup\mathbb{S} \in \mathbb{A}$.

In practice, $\mathbb{A}$ is the σ-algebra on $\mathcal{G}$ generated by the set of graphs containing a fixed edge (cfr. Appendix A.2).
We then define a (probability) measure μ for $\mathbb{A}$, by letting:

- for each edge e, $\mu(\mathcal{G}_e) = \mu(\{G : e \text{ is an edge of } G\}) = \frac{1}{2}$;

[7] In this case, it is somewhat easier to look at a tolerance space as an undirected graph, and use the standard graph-theoretic terminology. An element of the tolerance space will be therefore referred to as a "vertex", and an "edge" will be an unordered pair of vertices (i.e., a subset of the vertex set having cardinality 2).

- the two events $\mathcal{G}_e$ and $\mathcal{G}_{e'}$ are independent, whenever e and e' are different.

Now, μ is a standard probability (i.e., normalized) measure, corresponding to the construction of a random graph on ω where, for each pair of vertices, there is exactly a probability $\frac{1}{2}$ that they are joined by an edge.

Alternatively [Cam90], this could be rephrased as follows: consider a fixed enumeration of the edges (i.e., an enumeration $e_0, e_1, e_2, \dots$ of the set $\{e \subseteq \omega : |e| = 2\}$). Then, there is a bijective correspondence between $\mathcal{G}$ and the set of binary ω-sequences; we can thus consider $\{0,1\}^\omega$ as a probability space, by assuming that the event (set of sequences) having a "1" in a preassigned position is $\frac{1}{2}$.

In a more concrete (although a bit imprecise) way, we could think of a coin-tossing game: for each pair of vertices, we toss a (fair[8]) coin to decide whether to join them by an edge or not.

Now, the argument of the proof proceeds as follows. Let $A, B \subseteq_{\text{fin}} \omega$ be two finite disjoint sets of vertices (not both empty), and consider the following event[9]:

$$E(A,B) = \text{"there exists a vertex } x \text{ in } G \text{ which is adjacent to every vertex of } A \text{ and to no vertex of } B\text{".}$$

Now, let $x_0, x_1, \dots$ be an enumeration of the elements of $\omega \setminus (A \cup B)$, and consider, for each $n = 0, 1, \dots,$ the event:

$$F^n(A,B) = \text{"no one of the vertices } x_0, \dots, x_n \text{ satisfies the property of being adjacent to every vertex of } A \text{ and to no vertex of } B\text{".}$$

Clearly, $E(A,B)$ is the complement of the set $\cap_{n\in\omega} F^n(A,B)$, so (by Proposition A.2.1 (6)) $\mu(E(A,B)) = 1 - \mu(\cap_{n\in\omega} F^n(A,B))$. Observe that the sequence $F^n(A,B)$ satisfies the antimonotonic property, i.e., $F^{n+1}(A,B) \subseteq F^n(A,B)$ and thus, by Proposition A.2.1 (5),

$$\mu(\cap_{n\in\omega} F^n(A,B)) = \lim_{n\to\infty} \mu(F^n_G(A,B)).$$

But now, for each x, the probability of x to be non-adjacent to some vertex of A, or adjacent to some vertex of B, is one minus the probability of it to be adjacent to all vertices of A and non-adjacent to all vertices of B, i.e., $1 - 2^{-m}$ where $m = |A| + |B|$. Since this probability is independent for all x's, we have that

$$\mu(F^n(A,B)) = (1 - \frac{1}{2^m})^n$$

and so

$$\mu(\cap_{n\in\omega} F^n(A,B)) = \lim_{n\to\infty} (1 - \frac{1}{2^m})^n = 0$$

which finally means $\mu(E(A,B)) = 1$.

So $\mathbf{Prob}(E(A,B)) = \mu(E(A,B)) = 1$. Now, we are interested in the probability measure of the set of all graphs satisfying the saturation property, i.e.,

$$\mathbf{Prob}(\cap_{A,B} E(A,B)) = \mu(\cap_{A,B} E(A,B))$$

[8] Yet, as we shall see, fairness is not a strict requirement for the construction.

[9] We identify an event with the set of graphs $G \in \mathcal{G}$ satisfying the statement.

which is $1 - \mu(\cup_{A,B} E(A,B)^C)$. But $\cup_{A,B} E(A,B)^C$ is a countable union (because there are only countably many pairs of finite subsets of ω) of sets, each of them having measure $\mu(E(A,B)^C) = 1 - \mu(E(A,B)) = 0$. So, by Proposition A.2.1 (7), it also has measure zero, and thus the measure of the set of all universal homogeneous graphs is $1 - \mu(\cup_{A,B} E(A,B)^C) = 1 - 0 = 1$.

In other words, the random graph satisfies the stepwise-saturation property of Theorem 4.2.2 with probability 1. Thus, by the uniqueness property of Theorem 4.2.1 (using Property 4.3.1), the random graph is $\mathcal{T}_{\mathrm{R}}$ with probability 1.

As Cameron observes ([Cam90], Exercise 4.3.1), the whole proof does not really depend on the fairness assumption in the coin-tossing game. We could set the probability of each edge to be any (fixed) $p \in (0,1)$ and obtain the same result. In other words, the construction of Rado's graph is very robust, and does only depend on the important assumption that the choices made at each edge are independent: this is, as a matter of fact, the crux of the whole construction, because it gives a sufficient degree of freedom as to insure the universal property to hold.

4.6.2 Probabilistic construction of the universal gts

We are now ready to generalize the construction of Rado's tolerance space to the case of gts's; even though it would be possible to develop a machinery like that presented in the previous paragraph, we shall skip the details and come to the core of the proof without indulging in all the technicalities involved in it.

We want to build a random gts $\mathcal{T}_{RAND}$, much in the same way as we did for the random graph; take ω as underlying set. Now we have to construct the consistency predicate Con_{RAND}. In order to do this, consider a fixed enumeration $A_0, A_1, A_2, \ldots$ of the finite subsets of ω satisfying the following constraints:

1. the enumeration is injective; i.e., $A_i \neq A_j$ whenever $i \neq j$;

2. if $B \subset A_i$, then there exists $j < i$ such that $B = A_j$.

Such an enumeration can be built step-by-step by the following recursive procedure:

- **First step.** Set $A_0 = \emptyset$ and put $S = \omega$;

- **Inductive step.** Suppose that you have already built the subsequence $A_0, \ldots, A_k$, and let $n = \min S$. Then, for each $i = 0, \ldots, k$, define $A_{k+i+1} = A_i \cup \{n\}$; moreover, delete n from the set S.

This procedure produces the sequence $\emptyset$,$\{0\}$,$\{1\}$,$\{0,1\}$,$\{2\}$,$\{0,2\}$,$\{1,2\}$,$\{0,1,2\}$ etc.

Now, for each $i \in \omega$, decide whether $A_i \in \mathrm{Con}_{RAND}$ using the following randomized algorithm:

1. if $|A_i| \leq 1$, then put A_i in the consistency predicate (with probability 1);

2. otherwise, consider all the indices $j < i$ such that $A_j \subset A_i$: if there is an index for which $A_j \notin \mathrm{Con}_{RAND}$, then do not put A_i in the consistency predicate; otherwise, decide whether to put it in or not with probability $\frac{1}{2}$.

The procedure is defined in such a way that the resulting structure is necessarily a gts, because it is clearly downward-closed and contains all singletons.

Now, we have to prove that saturation holds with probability 1. Suppose that we have a finite sub-gts of $\mathcal{T}_{RAND}$, with underlying set $Y \subseteq_{\text{fin}} \omega$. Then, there will be a certain (finite) set of indices $\alpha_0 < \alpha_1 < \ldots < \alpha_k$ such that $\{A_{\alpha_i}, i = 0, \ldots, k\} = \wp(Y)$ (and, more precisely, $k = 2^{|Y|} - 1$), some of them corresponding to consistent sets. Let $\mathcal{C}$ be the set of subsets of Y which are in the consistency predicate Con_{RAND} (with $M = |\mathcal{C}|$ being the cardinality of such set), and let $\mathcal{A} \subseteq \mathcal{C}$ be a downward-closed subset of $\mathcal{C}$. We want to find an element $x \notin Y$ such that the only consistent subsets of $Y \cup \{x\}$ are those of the form $A \cup \{x\}$ for $A \in \mathcal{A}$.

For a fixed $x \notin Y$, there are exactly k non-singleton subsets of $Y \cup \{x\}$ containing x, and for each of them we have to decide whether it is consistent or not. Clearly, those sets A including x such that $A \setminus \{x\}$ is not consistent are, so to say, out of the game, because certainly (with probability 1) they will not be in the consistency predicate. We only have to worry about the sets of the form $B \cup \{x\}$ with $B \in \mathcal{C}$; for each of them, there is a probability $\frac{1}{2}$ that they are included (or not included) in the consistency predicate. So, the probability that x exactly satisfies our needs is just $\frac{1}{2^M}$.

Thus, the probability that *no x ever satisfies* saturation is

$$\lim_{n\to\infty}(1 - \frac{1}{2^M})^n$$

because the choice is made independently for each x; this limit is zero, as soon as $\mathcal{C} \neq \emptyset$ (which is clearly satisfied), and so $\mathcal{T}_{RAND}$ has the saturation property with probability 1.

Note that, also in this case, we could put the probability equal to some $p \in (0, 1)$ and still obtain the same result, like in the case of ordinary tolerance spaces.

4.6.3 Probabilistic construction of the universal MeES

We now want to establish a randomized construction for the universal homogeneous MeES, which we shall denote by $\mathcal{E}_{RAND}$. Take first ω as event set, and consider the same fixed enumeration $A_0, A_1, A_2, \ldots$ of $\wp_{\text{fin}}(\omega)$ as in paragraph 4.6.2. Now, we also need an enumeration $(e_0, B_0), (e_1, B_1), \ldots$ of $\omega \times \wp_{\text{fin}}(\omega)$ satisfying the following constraints

1. the enumeration is injective; i.e., if $i \neq j$ then either $e_i \neq e_j$, or $B_i \neq B_j$ (or both);

2. if $B \subset B_i$, then there is some $j < i$ such that $e_j = e_i$ and $B = B_j$.

This can be easily accomplished by first considering the standard (inverse) Cantor's coding for the set $\omega \times \omega$, namely $(\text{left}, \text{right}) : \omega \to \omega \times \omega$ as represented in the diagram:

	0	1	2	3	4	...
0	0	2	5	9	...	
1	1	4	8	...		
2	3	7	...			
3	6	...				
⋮	⋮					

Now, define, for each $i \in \omega$, $e_i = \mathrm{left}(i)$ and $B_i = A_{\mathrm{right}(i)}$. Injectivity follows from the fact that $\langle A_n \rangle_{n\in\omega}$ is injective; the other property follows from the analogous property of the A_i's, and from the fact that the projection of Cantor's coding along each row (or column) is a strictly increasing endofunction of ω.

We are then ready to express the random algorithm used to establish the consistency predicate Con_{RAND} and the enabling relation $\vdash_{RAND}$. The former is described as in Paragraph 4.6.2; for the latter, proceed as follows. First, form a binary sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$ with the pocedure defined, for each i, as follows:

1. if there is a $j < i$ such that $e_j = e_i$ and $\alpha_j = 1$, put $\alpha_i = 0$;
2. otherwise, put $\alpha_i = 1$ with probability $\frac{1}{2}$.

Following this procedure, for each $e \in \omega$, the set $\{i \in \omega : \alpha_i = 1 \wedge e_i = e\}$ is either empty or a singleton made by a unique index, say i_e. Put, for each set $B \in \mathrm{Con}_{RAND}$ and each $e \in \omega$

$$B \vdash_{RAND} e \text{ iff } i_e \text{ exists, } B_{i_e} \in \mathrm{Con}_{RAND} \text{ and } B_{i_e} \subseteq B.$$

Observe that this predicate is decidable: for each B, there will be an index j such that $B = B_j$, and so the subsequence $\alpha_0, \alpha_1, \ldots, \alpha_j$ is enough to decide whether $B \vdash_{RAND} e$ or not. Moreover, the structure is a MeES, and μ_e exists only if i_e exists and $B_{i_e} \in \mathrm{Con}_{RAND}$, and in that case $\mu_e = B_{i_e}$.

We should now prove the saturation property for $\mathcal{E}_{RAND}$. We follow the lines, and use the same notations, as in the proof of Theorem 4.3.5. For a fixed $y \notin Y$, the probability that y is consistent with all and only those subsets of Y which belong to $\mathcal{C}$ is just $\frac{1}{2^M}$, where $M = |\mathrm{Con}^Y_{RAND}|$.
Now, we have to consider the probability for y to be enabled in the ways described by the formula of page 60 (the proof for x is similar, and omitted): this is the probability of i_y to exist and $B_{i_y} = A$. We have to consider all indices j such that $e_j = y$: there will be a certain index $\hat{\jmath}$ s.t. $e_{\hat{\jmath}} = y$ and $B_{\hat{\jmath}} = A$. Now, the probability that $\alpha_{\hat{\jmath}} = 1$ is 1 minus the probability that $\alpha_j = 1$ for some $j < \hat{\jmath}$, or $j > \hat{\jmath}$. Clearly, there are $K = |\wp(A)| - 1$ indices j smaller that $\hat{\jmath}$ s.t. $e_j = y$, and the probability for one of them to be "1" is $1 - \frac{1}{2^K}$. Then, the probability that $\alpha_{\hat{\jmath}} = 1$ is precisely $\frac{1}{2^{K+1}}$.
So, the joint event we are interested in has probability $\frac{1}{2^{M(K+1)}}$ for a fixed y. The probability that y does not exists is thus

$$\lim_{n\to\infty} (1 - \frac{1}{2^{M(K+1)}})^n$$

which is 0. So y (and also x, for analogous reasons) exists with probability 1, whatever the choice of $Y \subseteq_{\mathrm{fin}} \omega$, $A \in \mathrm{Con}_{RAND}$ and $\mathcal{C} \subseteq \mathrm{Con}^Y_{RAND}$ (with $\mathcal{C} \neq \emptyset$).

This proves that $\mathcal{E}_{RAND}$ is isomorphic to the structure $\mathcal{E}_U$ of Theorem 4.3.5 with probability 1.

4.7 Summary of the universality results

In the following table, for convenience of the reader, we summarize the universality results obtained in this chapter. In each row of the table, we indicate a kind of representation, and the corresponding domain category. In the last two columns we give the reference to the universality result(s), both for representations and domains; an asterisk indicates that homogeneity was also proved.

Representations	Domains	Universality results for representations	Universality results for domains
tolerance spaces	coherent atomic dI-domains	*Thm. 4.3.1; *Subs. 4.6.1	*Cor. 4.3.1
generalized tolerance spaces	atomic dI-domains	*Thm. 4.3.3; *Subs. 4.6.2	*Cor. 4.3.2
min. enabling event struct.	dI-domains	*Thm. 4.3.5; *Subs. 4.6.3	Cor. 4.3.4
prime event structures, full trace automata	coherent dI-domains	—	Thm. 4.5.5

Chapter 5

Tolerance spaces and approximation

5.1 Introduction. Why tolerance spaces?

In this chapter, we aim at discussing in more detail the notion of tolerance space, in particular with respect to their application to measurement and approximation theory.

The importance of the concept of "tolerance" is nicely described by Shreyder in [Shr71], where he states that:

> The familiar concept of equivalence allows us to partition a set of objects into classes of "identical" objects from any particular point of view. However, in a number of cybernetic problems, it is more convenient to speak not of identical objects but rather of similar or indistinguishably different objects. For example, we may speak of points that are indistinguishably different for the eye, of words that are similar in meaning, of interchangeable members of a single collective, etc.

Zeeman [Zee62] introduced the name tolerance relation to mean those formal relations which are suitable to capture the essence of this definition of indistinguishability; later on, these relations were further studied in different contexts (by Kalmar [Kal67], in the description of a formal model of semantics; by Khalimsky et al. [KKM90], in the area of digital topology and computer graphics; by Shreyder himself, with applications to the theory of nonrigid taxonomy [Shr68a], and to the structure of collectives [Shr68b]). More recently, new insights in the study of tolerance relations were given by studies related to fuzzy analysis [TS95, Thi96b, Thi96a].

Clearly, the crucial difference between the definition of an equivalence relation and that of a tolerance relation is "transitivity". The question whether transitivity of indifference should be taken as an axiom or not has been largely discussed, since the works in psychometrics of the 50s.

A classical example against transitivity of indifference is the following, due to Luce [Luc56]: most people would prefer a cup of coffee with one spoon of sugar to a cup with five spoons; yet, if sugar were added to the first cup at the rate of 1/100 of gram, they would almost certainly be indifferent between successive cups. If indifference were transitive, they would have to be indifferent between the cup with one spoon and the cup with five spoons. So, indifference should not be taken to be transitive in the case of preferences, and many other examples of non-transitivity can also be found in many other practical situations. Does this also happen in the case of physical measurements? In the ideal situation where measuring instruments are perfect and always give a precise answer, the indifference relation should coincide with the identity. But the crude reality is that instruments are not infinitely precise, and the discriminating power of humans who use these instruments is limited: for these reasons, transitivity of indifference is a chimera also when measuring physical magnitudes.

Parikh [Par83] also presents many paradoxes originated by the false hope that transitivity holds in real-life situations. In particular, he quotes a famous puzzle (also discussed by Michael Dummett [Dum75] and Crispin Wright [Wri75]):

> Imagine a series of coloured patches, beginning with a red patch. Suppose, moreover, that the colour from patch to patch changes gradually so that the last patch is quite clearly not red. However, the change is so gradual that each patch is — to the eye — quite indistinguishable from the next in colour. Now [...] is there a last red patch in this series? Clearly there must be one since the series doesn't stay red forever. However, if there *is* such a last red patch, then the next one, indistinguishable from it in colour, is nonetheless *not* red. And surely that is absurd. Hence the paradox.

At this point, we think that the reader should be convinced of the importance of studying intransitive relations (i.e., tolerance relations) and apply them to the context of measurement, in particular in (but not with restriction to) the field of perception and social sciences.

5.2 An example — Analog-to-digital conversions and doublescales

In his paper [Smi93], Einar Smith presents an interesting example of how intransitivity of indifference arise naturally in physical measurement and computer science, and introduces a technical solution to this problem by defining the notion of "doublescale".

Consider the common problem of making an analog-to-digital conversion: in other words, suppose that a physical device must be designed to read the physical position of a pointer on an analog scale (i.e., a continuous scale) and to produce a digital (i.e., discrete) value in output. We can assume that the scale is divided into equally-spaced segments, each representing one digital value. Since the pointer has a non-negligible width, though, when it overlaps one of the segment borders, there will be no ways for the converter to decide which is the correct value to be output (see Fig. 5.1).

To bypass this problem, one could use two pointers, one displaced slightly from the other, and to choose in some logical way between the outputs available. Instead of using two displaced pointers, one could as well use one single pointer on two displaced scales, like in Fig. 5.2. As we can see, for example, when the pointer is in position A, no correct value can be read from the upper scale, but one correct (unambiguous) value is read from

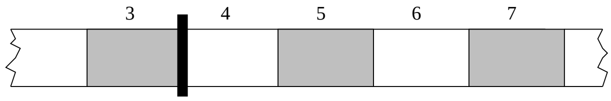


Figure 5.1: An analog scale with a pointer

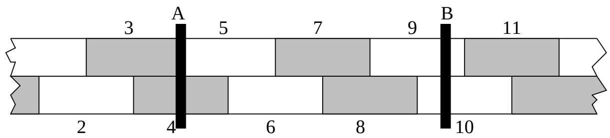


Figure 5.2: Two displaced scales

the lower scale (because the pointer is definitely in the segment 4). In general, for every possible position of the pointer, there is at least one unambiguous value read on one of the scales. In some cases, though, there are two correct values, like in position B, where the values 9 and 10 are both possible. Which is the correct one? which one should be output?

There is no answer to this question: as a matter of fact, we can say that both values are correct, because both are possible. In a sense, this means that the actual scenario is compatible with both of them; or, said otherwise, the two statements "the correct value is 9" and "the correct value is 10" are not contradictory, because of the (inherent) limitations in our resolution power.

This consideration has a strong impact on the way we should build our measuring scales; in fact, even though the magnitude we are measuring is of scalar type, there will be no linear (total) order on its values. Rather, we shall have a partial order, reflecting the impossibility of making comparisons on indistinguishable values. Looking at our two displaced scales, for example, we can certainly say that value 3 is "definitely smaller" than value 8, because the two values are incompatible (i.e., they are distinguishable by means of our measuring tool), but the two values 3 and 4 are "indistinguishable" in the sense that we have no ways to compare them. As a matter of fact, we can build two different converters in such a way that the same position of the pointer will be interpreted as 3 by the first converter, and as 4 by the second, due to a different internal logic used for choosing the correct value to output, when two values are possible.

In other words, our measuring scale will be the partial order represented in Fig. 5.3. In practice, this corresponds to the poset $(\mathbb{Z}, \ll)$ where

$$x \ll y \text{ if and only if } x + 1 < y.$$

We shall refer to this poset as the *doublescale*[1]. Later on, we shall introduce the notion of semiorder, and prove that the doublescale is an example of semiorder. In the present context, however, we content ourselves to observe that the incomparability relation of this semiorder is *not* an equivalence relation, but rather a tolerance relation. In Fig.5.4 this

[1]This is not the definition given in [Smi93], where a doublescale is taken to be any subposet of $(\mathbb{Z}, \ll)$ induced by an interval of $\mathbb{Z}$. However, any such doublescale is really a substructure of $(\mathbb{Z}, \ll)$, as [Smi93] observes in Example 4.11.

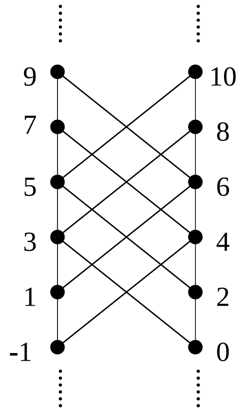


Figure 5.3: Hasse diagram of the doublescale

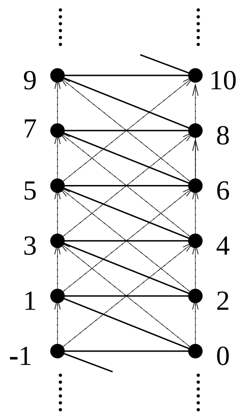


Figure 5.4: The co-doublescale

tolerance relation is shown: this is precisely the relation $\approx$ defined by

$$x \approx y \text{ if and only if } |x - y| \leq 1.$$

The tolerance space $(\mathbb{Z}, \approx)$ will be called *co-doublescale*, and will be denoted by $\mathcal{Z}$. Note that $\mathcal{Z}$, seen as a graph, is simply a bi-infinite path on countably many vertices (or, if you prefer, it is the symmetric reflexive closure of the covering relation associated to the standard ordering on the integers).

5.3 Tolerance spaces and continuous functions

In Chapter 4 we have considered the category **TolSp** of tolerance spaces[2] with embeddings; now, we shall discuss briefly the importance of tolerance-continuous maps, and thus

[2]In this chapter, the words "tolerance space" and "(undirected) graph" will often be used interchangeably; so we shall speak of "points" or "vertices" (nodes) to refer to the elements of the space (graph), and use the word "edge" (arc) to indicate a pair of co-related points. This confusion should not harm the reader.

introduce a supercategory of **TolSp**, where arrows are represented by continuous maps. As we shall prove, this category contains all limits of ω-chains, and we can give a nice direct (standard) construction for those (inverse) limits.

We let **cTolSp** be the category of (countable) tolerance spaces, with tolerance-continuous maps. As usual, unless otherwise stated, if $\mathcal{T}$ is a tolerance space, then X will represent the underlying set and co the tolerance relation. Moreover, if $\langle X_i \rangle_{i\in\omega}$ is an ω-sequence of sets, the elements of $\prod_{i\in\omega} X_i$ will be denoted by boldface letters, like $\mathbf{x}$, while x_i denotes the i-th element of $\mathbf{x}$.

Theorem 5.3.1 *The category* **cTolSp** *has inverse limits of ω-cochains.*

Proof: Let $(\mathcal{T}_n, f_n)_{n\in\omega}$ be an ω-cochain of tolerance spaces. Define the following data:

- X_∞ is the subset of $\prod_{n\in\omega} X_n$ containing only those vectors $\mathbf{x}$ such that, for all $n \in \omega$, $f_n(x_{n+1}) = x_n$;
- two vectors $\mathbf{x}, \mathbf{y} \in X_\infty$ are co_∞ related if and only if, for all $n \in \omega$, it holds that $x_n \,\mathrm{co}_n\, y_n$.

Let $\mathcal{T}_\infty$ be the corresponding tolerance space $(X_\infty, \mathrm{co}_\infty)$, and let, for each $n \in \omega$, $\psi_n : X_\infty \to X_n$ be defined as the n-th projection, i.e., $\psi_n(\mathbf{x}) = x_n$ for every vector $\mathbf{x} \in X_\infty$. Clearly, the ψ_n's are continuous maps. Moreover, for each $\mathbf{x} \in X_\infty$ we have $f_n(\psi_{n+1}(\mathbf{x})) = f_n(x_{n+1}) = x_n = \psi_n(\mathbf{x})$ as required for a cone. Let now $\mathcal{T}$ be a tolerance space, endowed with continuous functions $\phi_n : \mathcal{T} \to \mathcal{T}_n$ such that $f_n \circ \phi_{n+1} = \phi_n$. Define $h : \mathcal{T} \to \mathcal{T}_\infty$ as the function mapping $t \in \mathcal{T}$ to the sequence $h(t) = (\phi_0(t), \phi_1(t), \dots)$. This is well-defined, and moreover it is continuous: in fact, if $t \,\mathrm{co}\, t'$ then also $\phi_n(t) \,\mathrm{co}_n\, \phi_n(t')$ (by continuity of the ϕ_i's), and so also $h(t) \,\mathrm{co}_\infty\, h(t')$. Now, for each $t \in T$, one has $\psi_n(h(t)) = (h(t))_n = \phi_n(t)$, i.e., $\psi_n \circ h = \phi_n$, as required. Suppose finally that h' is another such function; then, for every $t \in T$, we should have $\psi_n(h'(t)) = \phi_n(t)$ which implies $h = h'$; so uniqueness is proved. $\Box$

We now introduce some more notation that we shall use in the sequel. If $(\mathcal{T}_n, f_n)_{n\in\omega}$ is an ω-cochain, we define, for each $n \le m$, the function $f_n^m : \mathcal{T}_m \to \mathcal{T}_n$ as the (continuous) function $f_n \circ f_{n+1} \circ \dots \circ f_{m-1}$ (by definition, f_n^n is taken to be the identity of $\mathcal{T}_n$). Observe that, if $\mathcal{T}_\infty$ is the inverse limit of the chain, then for all $\mathbf{x} \in \mathcal{T}_\infty$ and for any choice of $n \le m$, we have:

$$
\begin{aligned}
f_n^m(x_m) = f_n(f_{n+1}(\cdots f_{m-1}(x_m)\cdots)) &= \\
& f_n(f_{n+1}(\cdots f_{m-2}(x_{m-1})\cdots)) = \cdots = f_n(x_{n+1}) = x_n.
\end{aligned}
$$

An important point, here, is transitivity. A tolerance space is *transitive* if the tolerance relation is such; from a graph-theoretical point of view, a tolerance space is transitive if and only if every connected component is a clique (i.e., a complete graph). A very special case of transitivity happens when the tolerance relation is the identity (in which case we speak of a *totally disconnected* tolerance space).

One natural question is the following: suppose you have an ω-cochain of tolerance spaces, each of them being taken to represent a certain "resolution degree" of a measuring tool. Intransitivity happens because there are certain indistinguishable (but different) values the instrument can output. Of course, we can improve the resolution power of our instrument, i.e., take a more accurate one. In this case, we shall have in general more points and better approximations. The projection maps of the ω-cochain precisely represent the fact that each point in the (more accurate) tolerance space $\mathcal{T}_{n+1}$ is a better approximation of a point in the (less accurate) tolerance space $\mathcal{T}_n$; continuity serves to insure that we cannot introduce less accuracy (i.e., if we cannot distinguish two points at level $n+1$, we could not distinguish them at level n).

Yet, in the limit, we hope to have an infinite accuracy in the measurement, and therefore a transitive tolerance space. Is there some (formal) way to express this situation. In other words, is there a way of expressing a condition on the ω-cochain which is equivalent to its (inverse) limit being transitive?

In order to solve this problem, we introduce the notion of intransitive triple. An *intransitive triple* (or, a "V") of a tolerance space $\mathcal{T}$ is a triple $(x, y, z) \in X^3$ such that $x \operatorname{co} y$, $y \operatorname{co} z$ but $\neg(x \operatorname{co} z)$. We let $V(\mathcal{T})$ be the set of all intransitive triples of $\mathcal{T}$.
Notice that $\mathcal{T}$ is transitive precisely when it has no V's, i.e., when $V(\mathcal{T}) = \emptyset$. Also, trivially:

Property 5.3.1 *Let $f : \mathcal{T} \to \mathcal{T}'$ be continuous, and $(x, y, z) \in V(\mathcal{T})$; then either $(f(x), f(y), f(z)) \in V(\mathcal{T}')$ or it is a clique of $\mathcal{T}'$ (i.e., any two elements of the triple are co$'$-related).* □

Given an ω-cochain of tolerance spaces $(\mathcal{T}_n, f_n)_{n\in\omega}$, define, for each $n \in \omega$ and for each $(x, y, z) \in V(\mathcal{T}_n)$

$$\tau(x, y, z) = \sup\{k \geq n : \exists (x', y', z') \in V(\mathcal{T}_k).\ f_n^k(x') = x, f_n^k(y') = y, f_n^k(z') = z\}.$$

This is called the *resolution time* of (x, y, z). In practice, $\tau(x, y, z) = k < \infty$ means that, up to level k, there exist some V which is a better approximation of (x, y, z), but, as soon as we move to a higher level, this is no more true. Otherwise said, suppose we are doing a measurement using the n-th instrument, and we observe a "local contradiction" to transitivity (i.e., an intransitive triple). We can try to make better and better measurements, in order to solve our contradiction, but we know that this contradiction will be certainly solved not later then level k (maybe sooner, maybe not). If $\tau(x, y, z) = \infty$, then there is at least one case in which our contradiction will *never* be solved, and in principle we shall wait forever, making better and better experiments without ever succeeding.

We have then the following result:

Theorem 5.3.2 *Consider an ω-cochain $(\mathcal{T}_n, f_n)_{n\in\omega}$, and suppose that, for each $n \in \omega$ and $x \in X_n$, the set $f_n^{-1}(x)$ is finite. Then, the following are equivalent:*

1. *$\mathcal{T}_\infty$ is transitive;*
2. *for all $n \in \omega$ and all $(x, y, z) \in V(\mathcal{T}_n)$, one has $\tau(x, y, z) < \infty$.*

Proof: (1) $\Longrightarrow$ (2). Suppose that $\mathbf{x} \operatorname{co}_\infty \mathbf{y} \operatorname{co}_\infty \mathbf{z}$, and assume by contradiction that $\neg(x_n \operatorname{co}_n z_n)$ for some n. Then $(x_n, y_n, z_n) \in V(\mathcal{T}_n)$; let now $k = \tau(x_n, y_n, z_n) + 1$; by

definition of τ, there is no triple $(x', y', z') \in V(\mathcal{T}_k)$ which is mapped to (x_n, y_n, z_n) by f_n^k. But this means that (x_k, y_k, z_k) is not a V of $\mathcal{T}_k$, which implies that $x_k \operatorname{co}_k z_k$: this is impossible, because then, by continuity, we should have $x_n \operatorname{co}_n z_n$.
(2) $\Longrightarrow$ (1). Suppose that (2) does not hold, i.e., that there is some $n \in \omega$ and some $(x_n, y_n, z_n) \in V(\mathcal{T}_n)$ such that $\tau(x_n, y_n, z_n) = \infty$. This means that there exists an infinite sequence of V's $(x_n, y_n, z_n), (x_{n+1}, y_{n+1}, z_{n+1}), \dots$ such that $f_i(x_{i+1}) = x_i$ (for every $i > n$). Now, for each $i < n$, let $x_i = f_i^n(x_n)$, $y_i = f_i^n(y_n)$ and $z_i = f_i^n(z_n)$.
By construction, $\mathbf{x} = (x_0, x_1, \dots)$,$\mathbf{y} = (y_0, y_1, \dots)$ and $\mathbf{z} = (z_0, z_1, \dots)$ are elements of X_∞, and moreover $\mathbf{x} \operatorname{co}_\infty \mathbf{y} \operatorname{co}_\infty \mathbf{z}$, but $\neg(\mathbf{x} \operatorname{co}_\infty \mathbf{z})$, since $\neg(x_n \operatorname{co}_n z_n)$. □

If the conditions of the previous theorem are satisfied, we say that the cochain is *transitive* (in the limit). This does not mean, however, that each element of the cochain is transitive, neither that it is transitive from a certain index on.

In general, the transitivity condition is crucial if we want to use a chain of measuring instruments, but it is not enough. In fact, it does not guarantee the time we have to wait before a contradiction (a "V") is solved to be bounded in any sense. It could depend on the specific intransitive triple, or on the specific resolution level. For this reason, we introduce two more definitions.

For an ω-cochain $(\mathcal{T}_n, f_n)_{n\in\omega}$, and for each $n \in \omega$:

- define $\tau_n^* = \sup\{\tau(x, y, z) : (x, y, z) \in V(\mathcal{T}_n)\}$ (called the *resolution time at level n*);
- let $\tau^* = \sup_{n\in\omega}(\tau_n^* - n)$ (called the *global waiting time*).

We say that the cochain is *locally bounded transitive* (or lb-transitive) if it has finite resolution time at every level (i.e., if $\tau_n^* < \infty$ for all $n \in \omega$); it is *bounded transitive* (or b-transitive) if it has finite global waiting time (i.e., if $\tau^* < \infty$).

5.4 An example — The Negative Digit Representation for the reals

Consider the problem of representing real numbers using some setting which allows approximation to be taken into consideration. For sake of simplicity, let us simply limit ourselves to the unit interval $I = [0, 1]$: the most usual approach consists in representing each point of I using an infinite sequence of bits.

More precisely, we define a map[3]

$$\begin{array}{rlcl} \phi: & \{0,1\}^\omega & \to & I \\ & \mathbf{d} = d_1 d_2 \dots & \mapsto & \sum_{i=1}^\infty d_i 2^{-i} \end{array}$$

which is surjective, and thus allows one to represent any point of I with some infinite sequence. Note that ϕ is *not* a bijection, because each dyadic rational[4] has two representations (for example, the sequences $01111\dots$ and $10000\dots$ both represent the real number $1/2$).

[3] For each finite set Σ, we let Σ^* (Σ^ω, Σ^∞) be the set of finite (infinite, finite and infinite, resp.) strings over Σ. We also use the symbol $\preceq$ to denote the prefix relation between strings.

[4] A dyadic rational is a rational of the form $m/2^n$, with $m, n \in \mathbb{Z}$.

This representation is known as *positive digit representation in base 2* (or 2-PDR, for short). Of course, one can generalize this to any base $b > 1$, representing reals with sequences of digits from the set $\Gamma_b = \{0, 1, \ldots, b-1\}$, defining the representation map as

$$\begin{array}{rlcl} \phi : & \{0,1\}^\infty & \to & I \\ & \mathbf{d} = d_1 d_2 \ldots & \mapsto & \sum_{i=1}^{\infty} d_i b^{-i}. \end{array}$$

This is known as b-PDR; the usual way to write real numbers (the decimal notation) corresponds to the 10-PDR.

As observed by [Sün95], there are two problems with positive digit representations:

- first of all, they are not injective: some numbers have more than one representation (we have already noticed this in the binary case);
- secondly (and more importantly, as far as approximation is concerned) addition is not computable. To explain this seemingly surprising statement, we need introduce the notion of finite truncation. Every initial (finite) prefix of a sequence representing a real number can be thought of as an approximation to that number; to do this, one can extend the function ϕ to *finite* sequences, by putting:

  $$\begin{array}{rlcl} \phi : & \Gamma_b^* & \to & I \\ & \mathbf{d} = d_1 \ldots d_n & \mapsto & \sum_{i=1}^{n} d_i b^{-i}. \end{array}$$

  Now, a finite sequence can represent any (non-approximate) number of which it is a prefix, so, if $\preceq$ denotes the prefix relation among sequences, the sequence $\mathbf{d}$ is taken as representing the whole set $\{\mathbf{e} \in \Gamma_b^\omega : \mathbf{d} \preceq \mathbf{e}\}$, or, more precisely, the set of real numbers $\{\phi(\mathbf{e}) : \mathbf{e} \in \Gamma_b^\omega \wedge \mathbf{d} \preceq \mathbf{e}\}$. This is clearly a closed interval of real numbers. So, for example, the finite sequence $\mathbf{d} = 011010$ is mapped to:

  $$\phi(\mathbf{d}) = \frac{1}{4} + \frac{1}{8} + \frac{1}{32} = \frac{13}{32}$$

  and represents the closed interval $[\frac{13}{32}, \frac{14}{32}]$ (the left extreme is represented by the sequence $\mathbf{011010}0000\ldots$, the right extreme by $\mathbf{011010}1111\ldots$). As one can see, $\phi(\mathbf{d})$ is one extreme of the interval, and not — as one would probably like — the center.

  This has some unpleasant drawbacks: suppose you are given the approximate versions of two real numbers like, for instance:

  $$\mathbf{d}_1 = 0111 \qquad \mathbf{d}_2 = 0000$$

  and you are required to obtain (an approximate version of) their summation. Now:

  $$\begin{array}{rcl} \phi(\mathbf{d}_1) & = & \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{7}{16} \\ \phi(\mathbf{d}_2) & = & 0 \end{array}$$

  so $\mathbf{d}_1$ and $\mathbf{d}_2$ represent the intervals $A_1 = [\frac{7}{16}, \frac{1}{2}]$ and $A_2 = [0, \frac{1}{16}]$. Their summation is thus in the interval $B = [\frac{7}{16}, \frac{9}{16}]$: so, the result should be a truncated sequence

$\mathbf{e}$ corresponding to an interval C which includes B. Clearly, the only sequence satisfying this constraint is the empty one (which corresponds, by definition, to I), because even the 1-digit sequences 0 (corresponding to the interval $[0, 1/2]$) and 1 (corresponding to $[1/2, 1]$) are not precise.

So, in this case, even though we possess a 4-digit approximation of the two summands, we cannot give even a single digit approximation of the result.

From a more abstract viewpoint, the problem with positive digit representation is that it is not well-suited for dealing with approximation, in that approximation is taken to be "symmetric", i.e., the uncertainty interval represented by an approximated sequence should be centered in the value represented by the sequence.

To solve this problem, using an idea originated by Cauchy [Cau40] and extensively used in [Gia93] (see also [Bol95]), we must find a way of coding real numbers such that each digit may give a positive or a negative contribution: this introduces a sort of redundancy but allows to overcome the aforementioned difficulties ([Smy92], Example 5.0.13).

We now give a precise definition. For any integer $b > 1$ (called the "base"), we let $\Sigma_b = \{i \in \mathbb{Z} : |i| < b\}$ be the set of *signed digits* in base b. We define a function $\phi_b : \Sigma_b^\omega \to \mathbb{R}$ acting as follows:

$$\phi_b(\mathbf{d}) = \sum_{i=1}^{|\mathbf{d}|} d_i b^{-i}.$$

From now on, we suppose that a base $b > 1$ is fixed, and we omit the subscript b. Note that:

Lemma 5.4.1 *The image of Σ^ω w.r.t. ϕ is $[-1, 1]$.*

Proof: First observe that, for every $\mathbf{d} \in \Sigma^\omega$:

$$-(b-1)\sum_{i=1}^{\infty} b^{-i} \leq \phi(\mathbf{d}) \leq (b-1)\sum_{i=1}^{\infty} b^{-i}.$$

Since $\sum_{i=0}^{\infty} b^{-i} = \frac{b}{b-1}$, and thus $\sum_{i=1}^{\infty} b^{-i} = \frac{1}{b-1}$ one obtains:

$$-1 \leq \phi(\mathbf{d}) \leq 1.$$

For surjectivity, notice first that we just need to prove surjectivity on the interval $[0, 1]$, because if $\mathbf{d} = d_1 d_2 \dots$ is such that $\phi(\mathbf{d}) = \alpha$, then the sequence $\mathbf{e} = (-d_1)(-d_2)\dots$ is mapped to

$$\phi(\mathbf{e}) = \sum_{i=1}^{\infty} (-d_i) b^{-i} = -\phi(\mathbf{d}) = -\alpha.$$

Now, let $\alpha \in [0, 1)$ (the representation of 1 is given by the sequence $(b-1)(b-1)\dots$). We proceed building three sequences $\alpha_0, \alpha_1, \dots, r_0, r_1, \dots \in [0, 1]$ and $d_1, d_2, \dots \in \{0, 1, \dots, b-1\}$ such that, for all $i \geq 0$, $\alpha_i \in [0, b^{-i})$.

Inductively, let first $\alpha_0 = \alpha \in [0, 1)$ and $r_0 = 0$. Now, to assign $\alpha_{i+1}, r_{i+1}, d_{i+1}$, consider the intervals $I_k = [kb^{-(i+1)}, (k+1)b^{-(i+1)})$ for $k = 0, \dots, b-1$: they are disjoint and their union is $[0, b^{-i})$. So, by inductive hypothesis, there is (exactly) one k s.t. $\alpha_i \in I_k$; now let:

$$\begin{aligned} d_{i+1} &= k \\ r_{i+1} &= r_i + d_{i+1} b^{-(i+1)} \\ \alpha_{i+1} &= \alpha_0 - r_{i+1}. \end{aligned}$$

Consider the sequence $\mathbf{d} = d_1 d_2 \dots$:

$$\begin{aligned} \phi(\mathbf{d}) = \sum_{i=0}^{\infty} d_{i+1} b^{-(i+1)} &= \sum_{i=0}^{\infty} (r_{i+1} - r_i) = \\ &= \lim_{n\to\infty} \sum_{i=0}^{n} (r_{i+1} - r_i) = \lim_{n\to\infty} r_n = \\ &= \lim_{n\to\infty} (\alpha_0 - \alpha_n) = \alpha - \lim_{n\to\infty} \alpha_n. \end{aligned}$$

But $\alpha_n \in [0, b^{-n}]$, and so $\lim_{n\to\infty} \alpha_n = 0$. Thus $\phi(\mathbf{d}) = \alpha$, as required. □

Lemma 5.4.1 shows that we can represent every real number in the interval $[-1, 1]$ by means of an infinite sequence: as already noticed, this is not a one-to-one coding for the reals, but it is surjective. Clearly, every *finite* sequence should be correctly interpreted as a (closed) interval on the real line, as the following definition suggests.

We let $\psi : \Sigma^* \to \wp(\mathbb{R})$ be the function defined as follows:

$$\psi(\mathbf{d}) = [\phi(\mathbf{d}) - b^{-|\mathbf{d}|}, \phi(\mathbf{d}) + b^{-|\mathbf{d}|}].$$

Now, for each $n \geq 0$, let $T_n = \Sigma^n$ and co_n be a binary relation on T_n defined as follows:

$$\mathbf{d} \ \mathrm{co}_n \ \mathbf{e} \text{ iff } |\phi(\mathbf{d}) - \phi(\mathbf{e})| \leq 2b^{-n},$$

or, equivalently, if and only if $\psi(\mathbf{d}) \cap \psi(\mathbf{e}) \neq \emptyset$. Clearly, co_n is a tolerance relation (it is reflexive and symmetric), but it is not transitive in general: it corresponds to the incomparability relation of a semiorder (this is a direct consequence of the Scott-Suppes representation theorem [SS58]). We use $\mathcal{R}_n$ to denote the tolerance space (T_n, co_n). It is possible to give a direct interpretation of co_n using the notion of approximation, as done in the following:

Lemma 5.4.2 *For all n, and $\mathbf{d}, \mathbf{e} \in T_n$, we have that $\mathbf{d} \ \mathrm{co}_n \ \mathbf{e}$ iff there exist $\mathbf{d}', \mathbf{e}' \in \Sigma^\omega$ such that $\mathbf{d} \preceq \mathbf{d}'$, $\mathbf{e} \preceq \mathbf{e}'$ and $\phi(\mathbf{d}') = \phi(\mathbf{e}')$.*

Proof: Only if. Let $\alpha = \frac{b^n}{2}(\phi(\mathbf{d}) - \phi(\mathbf{e}))$. Clearly $\alpha \in [-1, 1]$. So, by Lemma 5.4.1, there exist $\mathbf{f}^+, \mathbf{f}^- \in \Sigma^\omega$ such that $\phi(\mathbf{f}^+) = \alpha$ and $\phi(\mathbf{f}^-) = -\alpha$. Let now:

$$\begin{aligned} \mathbf{d}' &= d_1 d_2 \dots d_n f_1^- f_2^- \dots \\ \mathbf{d}' &= e_1 e_2 \dots e_n f_1^+ f_2^+ \dots . \end{aligned}$$

We then have:

$$\phi(\mathbf{d}') = \phi(\mathbf{d}) + \phi(\mathbf{f}^-) b^{-n} = \phi(\mathbf{d}) - \alpha b^{-n}$$

and

$$\phi(\mathbf{e}') = \phi(\mathbf{e}) + \phi(\mathbf{f}^+)b^{-n} = \phi(\mathbf{e}) + \alpha b^{-n}.$$

But $2\alpha b^{-n} = \phi(\mathbf{d}) - \phi(\mathbf{e})$ which implies $\phi(\mathbf{d}) = 2\alpha b^{-n} + \phi(\mathbf{e})$ and so

$$\phi(\mathbf{d}') = 2\alpha b^{-n} + \phi(\mathbf{e}) - \alpha b^{-n} = \phi(\mathbf{e}) + \alpha b^{-n} = \phi(\mathbf{e}').$$

If. We have that:

$$\phi(\mathbf{d}) - \phi(\mathbf{e}) = \phi(\mathbf{d}') - \phi(d_{n+1}d_{n+2}\dots)b^{-n} - \phi(\mathbf{e}') + \phi(e_{n+1}e_{n+2}\dots)b^{-n}.$$

Since $\phi(e_{n+1}e_{n+2}\dots) - \phi(d_{n+1}d_{n+2}\dots) \in [-2,2]$ we have that:

$$|\phi(\mathbf{d}) - \phi(\mathbf{e})| \leq 2b^{-n}.$$

□

The above Lemma 5.4.2 could be rephrased by saying that two n-truncated approximations are incomparable via co_n precisely when they are prefixes of two infinite sequences representing the same real number.

An immediate consequence of Lemma 5.4.2 is that the function $f_n : T_{n+1} \to T_n$ defined by

$$f_n(d_1 \dots d_n d_{n+1}) = d_1 \dots d_n$$

is continuous:

Corollary 5.4.1 *The map $f_n : \mathcal{R}_{n+1} \to \mathcal{R}_n$ is continuous.* □

In other words, $(\mathcal{R}_n, f_n)_{n\in\omega}$ is an ω-cochain of tolerance spaces. Now, we prove that it is bounded transitive. We first need the following

Lemma 5.4.3 *If $\mathbf{x}, \mathbf{y} \in T_n$ then $|\phi(\mathbf{x}) - \phi(\mathbf{y})|$ is an integer multiple of b^{-n}. Thus, $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in V(T_n, \mathrm{co}_n)$ if and only if there exist some integers k, h such that $|k| \leq 2$, $|h| \leq 2$ and $|k + h| > 2$ for which*

$$\begin{aligned} \phi(\mathbf{x}) + kb^{-n} &= \phi(\mathbf{y}) \\ \phi(\mathbf{y}) + hb^{-n} &= \phi(\mathbf{z}). \end{aligned}$$

Proof: For the first part, we have

$$b^n(\phi(\mathbf{x}) - \phi(\mathbf{y})) = b^n \sum_{i=1}^{n} (x_i - y_i)b^{-i} = \sum_{i=1}^{n} (x_i - y_i)b^{n-i}$$

which is an integer. Now, for the second part, by Lemma 5.4.2, we have:

$$|\phi(\mathbf{x}) - \phi(\mathbf{y})| \leq 2b^{-n} \implies \phi(\mathbf{x}) + kb^{-n} = \phi(\mathbf{y})$$

for a suitable choice of $k \leq 2$. The same holds for $\mathbf{y}$ and $\mathbf{z}$, but

$$|\phi(\mathbf{x}) - \phi(\mathbf{z})| > 2b^{-n}$$

and thus

$$|\phi(\mathbf{x}) - \phi(\mathbf{z})| = |\phi(\mathbf{x}) - \phi(\mathbf{y}) + \phi(\mathbf{y}) - \phi(\mathbf{z})| = | - kb^{-n} - hb^{-n}| = |k+h|b^{-n}$$

and this concludes the proof by showing that $|k+h| > 2$. □

We can now prove that the chain is bounded transitive with global waiting time 0 whenever the base b is greater than 2.

Theorem 5.4.1 *If $b > 2$, then the chain $(\mathcal{R}_n, f_n)_{n\in\omega}$ is bounded transitive with global waiting time $\tau^* = 0$.*

Proof: We just have to show that every V has no counterimages which are still V's. Suppose, by contradiction, that $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in V(T_n, \mathrm{co}_n)$ and also $(\mathbf{x}\alpha, \mathbf{y}\beta, \mathbf{z}\gamma) \in V(T_{n+1}, \mathrm{co}_{n+1})$, for some choice of $\alpha, \beta, \gamma \in \Sigma$. Then, by Lemma 5.4.3, there are $k, h, k', h' \in \{-2, \ldots, 2\}$ such that $|k+h| > 2$, $|k'+h'| > 2$ and

$$\begin{array}{rcl} \phi(\mathbf{x}) + kb^{-n} & = & \phi(\mathbf{y}) \\ \phi(\mathbf{y}) + hb^{-n} & = & \phi(\mathbf{z}) \\ \phi(\mathbf{x}\alpha) + k'b^{-(n+1)} & = & \phi(\mathbf{y}\beta) \\ \phi(\mathbf{y}\beta) + h'b^{-(n+1)} & = & \phi(\mathbf{z}\gamma). \end{array}$$

This implies:

$$\left\{ \begin{array}{l} \phi(\mathbf{x}) + \alpha b^{-(n+1)} + k'b^{-(n+1)} = \phi(\mathbf{y}) + \beta b^{-(n+1)} \\ \phi(\mathbf{y}) + \beta b^{-(n+1)} + h'b^{-(n+1)} = \phi(\mathbf{z}) + \gamma b^{-(n+1)} \end{array} \right.$$

and thus

$$\left\{ \begin{array}{l} \phi(\mathbf{x}) + (\alpha + k')b^{-(n+1)} = \phi(\mathbf{x}) + kb^{-n} + \beta b^{-(n+1)} \\ \phi(\mathbf{y}) + (\beta + h')b^{-(n+1)} = \phi(\mathbf{y}) + hb^{-n} + \gamma b^{-(n+1)} \end{array} \right.$$

$$\left\{ \begin{array}{l} (\alpha - \beta + k')b^{-(n+1)} = kb^{-n} \\ (\beta - \gamma + h')b^{-(n+1)} = hb^{-n} \end{array} \right.$$

$$\left\{ \begin{array}{l} \alpha - \beta + k' = kb \\ \beta - \gamma + h' = hb. \end{array} \right.$$

Adding these equations memberwise, we obtain

$$\alpha - \gamma + k' + h' = (k+h)b.$$

Its right-hand side has absolute value at least $3b$ (because $|k+h| > 2$), while its left-hand side is clearly at most $|\alpha - \gamma| + |k' + h'|$, which is not greater than $2b - 2 + 4 = 2b + 2$ (because $|k' + h'| \leq 4$). So, for equality to hold, one should have $3b \leq 2b + 2$ which implies $b \leq 2$, contradicting the hypothesis $b > 2$. □

Notice that Theorem 5.4.1 is not true when $b = 2$ as the following easy example shows. The triple (0 0,1 0,1 1) is a V of $\mathcal{R}_2$ (they represent the values 0, 1/2, 3/4 respectively). Yet, the triple (0 0 1,1 0 −1,1 1 −1) is still a V of $\mathcal{R}_3$ (the values are, in this case, 1/8, 3/8, 5/8). Nevertheless, the chain is bounded transitive also in the case $b = 2$, but with resolution time 1:

Theorem 5.4.2 *If $b = 2$, then the chain $(\mathcal{R}_n, f_n)_{n\in\omega}$ is bounded transitive with global waiting time $\tau^* = 1$.*

Proof: The proof is analogous to that of Theorem 5.4.1. Suppose that $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a V of (T_n, co_n), $(\mathbf{x}\alpha, \mathbf{y}\beta, \mathbf{z}\gamma) \in V(T_{n+1}, \mathrm{co}_{n+1})$ and $(\mathbf{x}\alpha\alpha', \mathbf{y}\beta\beta', \mathbf{z}\gamma\gamma') \in V(T_{n+2}, \mathrm{co}_{n+2})$. Then, we have:

$$\begin{aligned}
\phi(\mathbf{x}) + kb^{-n} &= \phi(\mathbf{y}) \\
\phi(\mathbf{y}) + hb^{-n} &= \phi(\mathbf{z}) \\
\phi(\mathbf{x}\alpha) + k'b^{-(n+1)} &= \phi(\mathbf{y}\beta) \\
\phi(\mathbf{y}\beta) + h'b^{-(n+1)} &= \phi(\mathbf{z}\gamma) \\
\phi(\mathbf{x}\alpha\alpha') + k''b^{-(n+2)} &= \phi(\mathbf{y}\beta\beta') \\
\phi(\mathbf{y}\beta\beta') + h''b^{-(n+2)} &= \phi(\mathbf{z}\gamma\gamma').
\end{aligned}$$

for some k, h, k', h', k'', h'' satisfying the constraints. This means:

$$\begin{cases} \phi(\mathbf{x}) + (\alpha - \beta)b^{-(n+1)} + (\alpha' - \beta' + k'')b^{-(n+2)} = \phi(\mathbf{y}) \\ \phi(\mathbf{y}) + (\beta - \gamma)b^{-(n+1)} + (\beta' - \gamma' + h'')b^{-(n+2)} = \phi(\mathbf{z}) \end{cases}$$

which implies

$$\begin{cases} (\alpha - \beta)b^{-(n+1)} + (\alpha' - \beta' + k'')\beta^{-(n+2)} = kb^{-n} \\ (\beta - \gamma)b^{-(n+1)} + (\beta' - \gamma' + h'')\beta^{-(n+2)} = hb^{-n}. \end{cases}$$

Adding these equations memberwise, we obtain:

$$(\alpha - \gamma)b^{-(n+1)} + (\alpha' - \gamma' + k'' + h'')b^{-(n+2)} = (k + h)b^{-n}$$

$$(\alpha - \gamma)b + \alpha' - \gamma' + k'' + h'' = (k + h)b^2$$

$$\alpha' - \gamma' + k'' + h'' = (k + h)b^2 - (\alpha - \gamma)b.$$

Substituting $b = 2$:

$$\alpha' - \gamma' + k'' + h'' = 4(k + h) - 2(\alpha - \gamma),$$

and an exhaustive case analysis shows that this is a contradiction. □

Clearly, this is not the only way one can use to prove that co_∞ is transitive, neither the more direct one, but it gives us much more information, because it tells us that not only transitivity is reached, but every "local" counterexample to transitivity is resolved after only one step (two steps at most, in the binary case).

The same results hold for the usual positive digit representation, even though the situation trivializes in that case: it turns out, as one can easily see, that

- for every n, the elements of T_n are mapped injectively to the unit interval by ϕ (i.e., unlike in the negative digit representation, we do not have multiple finite sequences of the same length corresponding to the same number);
- the approximation interval induced by a finite sequence $\mathbf{x} \in T_n$ is a closed interval of length b^{-n} whose left extreme is $\phi(\mathbf{x})$; thus, two such intervals intersect in at most one point;

- the previous observation implies that (T_n, co_n) is simply the graph P_{b^n} (the path with b^n nodes), and the projection f_n simply shrinks every subpath of length b to a single node;

- as a consequence, the only V's of (T_n, co_n) are those triples corresponding to three consecutive nodes in the path, which can never be image of any V at the next level; this means that the global resolution time in this case is $\tau^* = 0$, regardless of the base used for the representation.

We shall return to the negative digit representation later on, proving that it is possible to completely re-construct the usual (Euclidean) topology of the unit interval by means of an approximation cochain of finite tolerance spaces.

5.5 Approximation sequences and inverse images of topologies

In the previous sections, we have tried to convince the reader about the importance and usefulness of ω-cochains of tolerance spaces for describing a complex space (in the previous example: the set of real numbers) via an increasing set of approximating (finite) spaces. Of course, what we were missing until now was a way of obtaining the topological structure of the limit object as a limit of topologies on the approximations. In this section, we shall pave our way towards the definition of a topology on the limit, by using the notion of inverse image of a topology.

We shall be considering tolerance spaces on which a topology is defined, and call them *topological tolerance spaces*. When no confusion arises, if $\mathcal{T}$ is a topological tolerance space, we let Ω (possibly with self-explaining sub- or superscripts) denote the topology. From now on, we shall use the words *approximation (topological) sequence* to indicate an ω-cochain in the category **cTolSp** (and the adjective "topological" is added if each tolerance space is also topological).

First, consider the following result, which is a slight generalization of Theorem I.13.2 of [Gaa64]:

Theorem 5.5.1 *Let I be an index set and $(f_i : X \to Y_i)_{i \in I}$ be a family of functions. Suppose that, for each $i \in I$, there is a topology Ω_i defined on Y_i. Then, there is a weakest topology Ω on X such that the f_i's are (topologically) continuous; Ω is generated by the subbase*

$$\mathcal{S} = \{f_i^{-1}(O) : i \in I, O \in \Omega_i\}.$$

Proof: First, observe that at least one such topology exists (the discrete topology on X). Note also that $\mathcal{S}$ is in fact a subbase of a topology on X; in fact, if $x \in X$, take any $i \in I$ and consider $f_i(x)$: this is a point of Y_i, and clearly, taking any open set $O \in \Omega_i$ such that $f_i(x) \in O$, we have $x \in f_i^{-1}(O)$; so, there is an $S \in \mathcal{S}$ such that $x \in S$, as required. Let now Ω be the topology generated by $\mathcal{S}$. Clearly, every function f_i is continuous with respect to this topology. Moreover, if Ω' is another such topology, then $\mathcal{S} \subseteq \Omega'$, and so $\Omega \subseteq \Omega'$: this proves minimality. $\square$

The topology Ω of Theorem 5.5.1 is called the *inverse image* of the family $(\Omega_i)_{i\in I}$ under $(f_i)_{i\in I}$. In the special case $|I| = 1$, the theorem can be rephrased in the following, more canonical, form:

Corollary 5.5.1 *If $f : X \to Y$ is a function and Ω is a topology on Y, then there is a weakest topology $f^{-1}(\Omega)$ on X such that f is continuous. In particular, a set is open in $f^{-1}(\Omega)$ if and only if it is the counterimage of some open set in Ω.*

Proof: We just have to show that, in this case, the set $\mathcal{S} = \{f_i^{-1}(O) : i \in I, O \in \Omega_i\}$ is in fact itself a topology. Clearly, $\emptyset = f^{-1}(\emptyset) \in \mathcal{S}$ and $X = f^{-1}(Y) \in \mathcal{S}$. Moreover, for any two open sets $O_1, O_2 \in \Omega$, one has $f^{-1}(O_1 \cap O_2) = f^{-1}(O_1) \cap f^{-1}(O_2)$. Finally, if $(O_i)_{i\in I}$ is a family of open sets, then $f^{-1}(\cup_{i\in I} O_i) = \cup_{i\in I} f^{-1}(O_i)$. □

The following result states that the technique of inverse image topology transforms also the inverse limits in the category **cTolSp** into inverse limits in the category **Top** of topological spaces with continuous functions[5].

Theorem 5.5.2 *Let $(\mathcal{T}_i, f_i)_{i\in\omega}$ be a topological approximation sequence. Moreover, suppose that $\mathcal{T}_\infty$ is the inverse limit of the sequence, with projections $\pi_i : \mathcal{T}_\infty \to \mathcal{T}_i$. The topology Ω_∞ which is the inverse image of the family $(\Omega_i)_{i\in\omega}$ under $(\pi_i)_{i\in\omega}$ makes $\mathcal{T}_\infty$ into a topological space which is also the inverse limit of the approximation sequence considered as an ω-cochain in the category* **Top**.

Proof: We already know that $f_i \circ \pi_{i+1} = \pi_i$, by Theorem 5.3.1. Now, suppose that (X, Ω) is another topological space, endowed with a family $\phi_i : X \to X_i$ of continuous maps such that $f_i \circ \phi_{i+1} = \phi_i$. Define $h : X \to X_\infty$ by putting

$$h(x) = \langle \phi_i(x) \rangle_{i\in\omega}.$$

Now, $\pi_i(h(x)) = (h(x))_i = \phi_i(x)$, so $\pi_i \circ h = \phi_i$, as required. Moreover, h is continuous. In fact, let Ω' be the inverse image topology of the family $(\Omega_i)_{i\in\omega}$ under $(\phi_i)_{i\in\omega}$: clearly, $\Omega' \subseteq \Omega$, because Ω' is the weakest topology for which the ϕ_i's are continuous. In order to prove that h is continuous, using Lemma A.3.1, we just need to show that $h^{-1}(\pi_i^{-1}(O)) \in \Omega$ for all $i \in \omega$ and $O \in \Omega_i$. But

$$\begin{aligned} h^{-1}(\pi_i^{-1}(O)) = \{x \in X : h(x) \in \pi_i^{-1}(O)\} = \\ = \{x \in X : \pi_i(h(x)) \in O\} = \{x \in X : \phi_i(x) \in O\} = \phi_i^{-1}(O) \end{aligned}$$

and $\phi_i^{-1}(O)$ is an element of the standard subbase for Ω', and thus $\phi_i^{-1}(O) \in \Omega$ also. □

In other words, given a topological approximation sequence $(\mathcal{T}_i, f_i)_{i\in\omega}$, there is a standard way to topologize the limit of a sequence, and obtain a topological tolerance space which is indeed the limit of the corresponding cochain both in the category of tolerance spaces and in the category of topological spaces.

[5]When dealing with topological tolerance spaces, the use of the word "continuous" may be ambiguous (because we do not specify whether we are considering tolerance or topological continuity). To solve this problem, we usually put an explaining adjective near the word; nevertheless, when no confusion arises (or when a function is *both* topologically and tolerance continuous) we shall not indulge in any further explanation.

An important case of the previous construction happens when all the topologies Ω_i are discrete (i.e., $\Omega_i = \wp(X_i)$ for every i). In this case, we say that the sequence is *discretely topologized.*

Property 5.5.1 *Let $(\mathcal{T}_i, f_i)_{i\in\omega}$ be a discretely topologized approximation sequence. Then, the limit topology Ω_∞ has the set*

$$\mathcal{S}_\infty = \{\pi_i^{-1}(x) : i \in \omega \text{ and } x \in X_i\}$$

as base.

Proof: We already know that a base for Ω_∞ is given by the set $\mathcal{S} = \{\pi_i^{-1}(O) : i \in \omega \text{ and } O \in \Omega_i\}$, and clearly $\mathcal{S}_\infty \subseteq \mathcal{S}$. Moreover, every open set $O \in \Omega_i$ is the union of singleton open sets, and $\pi_i^{-1}(O) = \cup_{x\in O}\pi_i^{-1}(x)$. Thus $\mathcal{S}_\infty$ is really a subbase of Ω_∞. Now, to prove that this is a base, suppose that $\mathbf{x} \in \pi_i^{-1}(y) \cap \pi_j^{-1}(z)$ with $i, j \in \omega$, $y \in X_i$ and $z \in X_j$. This means that $\pi_i(\mathbf{x}) = x_i = y$ and $\pi_j(\mathbf{x}) = x_j = z$; suppose, without loss of generality, that $i \leq j$. Then, since $f_i^j(x_j) = x_i$, we have that $f_i^j(z) = y$. So, if an infinite sequence belongs to $\pi_j^{-1}(z)$, it must have a y in the i-th position, and thus it also belongs to $\pi_i^{-1}(y)$. In other words, $\pi_j^{-1}(z) \subseteq \pi_i^{-1}(y)$. This way, we have $\mathbf{x} \in \pi_j^{-1}(z) = \pi_j^{-1}(z) \cap \pi_i^{-1}(y)$. □

5.6 Reduction of tolerance spaces and direct images of topologies

In this section, we shall briefly discuss another well-known technique for inducing topologies, and use it to define the quotient topological space relative to a reduced tolerance space.

The following, which dualizes Corollary 5.5.1, is a standard theorem of general topology:

Theorem 5.6.1 (Gaal [Gaa64], Theorem I.13.3) *If $f : X \to Y$ is a function and Ω is a topology on X, then there is a strongest topology $f(\Omega)$ on Y such that f is continuous. In particular, a set is open in $f(\Omega)$ if and only if its inverse image is open in Ω.* □

By using this result, one can introduce the notion of quotient space. Let (X, Ω) be a topological space and R be an equivalence relation on X; there is a natural map $\psi_R : X \to X/R$ which maps every element x of X to the equivalence class $[x]_R$. Now, one can take the $\psi_R(\Omega)$, which is the strongest topology on X/R for which ψ_R is continuous. The topological space $(X/R, \psi_R(\Omega))$ is called the *quotient space* of (X, Ω) with respect to R, and the topology $\psi_R(\Omega)$ is usually denoted by Ω/R.

The notion of quotient space is very useful, and corresponds to the process of "identifying" some points in a topological space. For example, if one considers the unit interval [0,1] (as a subspace of $\mathbb{R}$, with the Euclidean topology), and identifies 0 and 1 (i.e., takes the least equivalence relation R on $[0, 1]$ for which $0 \, R \, 1$), the quotient space obtained is (homeomorphic to) the circle S^1.

We shall apply the above introduced concepts to the special case of reduced tolerance spaces. Let $\mathcal{T}$ be a tolerance space; for each $x \in X$, define $\mathrm{co}(x) = \{y \in X : x \,\mathrm{co}\, y\}$, and let

$$x \approx y \text{ iff } \mathrm{co}(x) = \mathrm{co}(y).$$

We have that

Lemma 5.6.1 *The relation $\approx$ is an equivalence relation; moreover, if $x \approx x'$ and $y \approx y'$ then*

$$x \,co\, y \iff x' \,co\, y'.$$

Proof: It is clear that $\approx$ is an equivalence relation. Now $x \approx x'$ precisely means that, for all z, $x \,\mathrm{co}\, z \iff x' \,\mathrm{co}\, z$. So we have $x \,\mathrm{co}\, y \iff x' \,\mathrm{co}\, y \iff x' \,\mathrm{co}\, y'$. □

Thus, we can define a new tolerance space $\mathcal{T}^*$, whose underlying set is $X/\approx$ and with tolerance relation co^* described by putting $[x]\,\mathrm{co}^*\,[y]$ if and only if $x \,\mathrm{co}\, y$ (this is well defined, in virtue of Lemma 5.6.1). The space $\mathcal{T}^*$ is called the *reduction* of $\mathcal{T}$, and we say that $\mathcal{T}$ is *reduced* if and only if it is isomorphic to its own reduction, i.e., $\mathcal{T} \cong \mathcal{T}^*$. Clearly:

Property 5.6.1 *The following properties of a tolerance space $\mathcal{T}$ are equivalent:*

1. *$\mathcal{T}$ is transitive;*
2. *$\mathcal{T}^*$ is totally disconnected.* □

Now, if we have a topological tolerance space $\mathcal{T}$, we can consider the quotient topology $\Omega^* = \Omega/\approx$ on the reduction $\mathcal{T}^*$: the so-obtained topology is often called the *reduced topology* (and the corresponding space is called the "reduced (topological) tolerance space"). The following result will be very useful in the sequel:

Theorem 5.6.2 *Let $\mathcal{T} = (X, co, \Omega)$ be a topological tolerance space, and let (Y, Ω') be another topological space. Suppose that $f : X \to Y$ is a function such that:*

1. *f is a surjective, continuous map;*
2. *for any two $x, y \in X$, one has $x \approx y$ if and only if $f(x) = f(y)$.*

Then, there is a continuous bijection $h : (X^*, \Omega^*) \to (Y, \Omega')$; moreover, if f is open, then h is a homeomorphism.

Proof: Define a function $h : X^* \to Y$ by letting $h([x]_\approx) = f(x)$. This is well-defined, because $x \approx y$ implies $f(x) = f(y)$. Moreover, h is injective (because $f(x) = f(y)$ implies $x \approx y$), and surjective (because f is). Now, let $\psi : X \to X^*$ be the natural map $x \mapsto [x]_\approx$. Remember that (Theorem 5.6.1) $O \in \Omega^*$ if and only if $\psi^{-1}(O) \in \Omega$. So, suppose $O \in \Omega'$: $h^{-1}(O)$ is open if and only if $\psi^{-1}(h^{-1}(O))$ is open, which is equivalent to saying that $\{x \in X : f(x) \in O\} = f^{-1}(O)$ is open. But f is continuous, so we conclude that h is continuous also. For the continuity of the inverse (in the case that f is open), suppose that O is open in Ω^*, i.e., $\psi^{-1}(O) \in \Omega$. Now $f(\psi^{-1}(O)) = f(O)$ which is open, because f is an open map. □

As a final observation, note that, in the statement of Theorem 5.6.2, one can substitute $x \approx y$ with $x \,\mathrm{co}\, y$, whenever $\mathcal{T}$ is transitive (because in that case $\approx$ coincides with co).

5.7 Approximating the Cantor set

In this section, we shall apply the techniques developed for topologizing the limits of approximation sequences to the case of positive digit representations of real numbers in the unit interval. In particular, we will show that the reduced limit topology is related with the standard Euclidean topology of $[0,1]$, when discrete topologies are taken on the approximation sequence: more precisely, we shall obtain a space which is homeomorphic to the standard Cantor set.

We first introduce some notation, which is similar to that used for negative-digit representations, with the only difference that here we are going to represent (approximations of) real numbers by (finite) sequences of unsigned bits. For every $n \in \omega$, we let R_n denote the set $\{0,1\}^n$, and use R_∞ for $\{0,1\}^\omega$ (the set of infinite binary sequences). Moreover, as usual, we use $\preceq$ to mean the prefix relation on the set $\{0,1\}^\infty$ of (finite and infinite) sequences of bits. For sake of simplicity, we assume that the indexes corresponding to each sequence start from 1.

We define a function:

$$\begin{array}{llll} \varphi: & \{0,1\}^\infty & \to & \mathbb{R} \\ & \mathbf{x} & \mapsto & \sum_{i=1}^{|\mathbf{x}|} \frac{x_i}{2^i}. \end{array}$$

mapping each sequence to a real number. We have that:

Property 5.7.1 *For all $\mathbf{x} \in \{0,1\}^\omega$, we have $\varphi(\mathbf{x}) \in [0,1]$. Moreover, for every $\alpha \in [0,1]$ there exists an infinite sequence $\mathbf{x} \in \{0,1\}^\omega$ such that $\varphi(\mathbf{x}) = \alpha$.*

Proof: Similar to Lemma 5.4.1. □

Now, for each $n \in \omega$, one can define a tolerance relation co_n^R on R_n, by putting:

$$\mathbf{x}\,\mathrm{co}_n^R\,\mathbf{y} \text{ iff } |\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq 2^{-n}.$$

The tolerance space $\mathcal{R}_n = (R_n, \mathrm{co}_n^R)$, seen as an undirected graph, is simply a path of 2^n vertices. Moreover, we define for all $n \in \omega$, a map $f_n : R_{n+1} \to R_n$ as follows

$$f_n(x_1 x_2 \dots x_{n+1}) = x_1 x_2 \dots x_n.$$

Observe that:

Lemma 5.7.1 *The function $f_n : \mathcal{R}_{n+1} \to \mathcal{R}_n$ is (tolerance) continuous.*

Proof: Suppose that $\mathbf{x}\,\mathrm{co}_{n+1}^R\,\mathbf{y}$; then:

$$\begin{aligned} |\varphi(f_n(\mathbf{x})) - \varphi(f_n(\mathbf{y}))| &= \left| \sum_{i=1}^{n} \frac{x_i}{2^i} - \sum_{i=1}^{n} \frac{y_i}{2^i} \right| = \\ &= \left| \sum_{i=1}^{n+1} \frac{x_i}{2^i} - \frac{x_{n+1}}{2^{n+1}} - \sum_{i=1}^{n+1} \frac{y_i}{2^i} + \frac{y_{n+1}}{2^{n+1}} \right| = \left| \varphi(\mathbf{x}) - \varphi(\mathbf{y}) + \frac{y_{n+1} - x_{n+1}}{2^{n+1}} \right| \leq \\ &\leq |\varphi(\mathbf{x}) - \varphi(\mathbf{y})| + \frac{1}{2^{n+1}} \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n} \end{aligned}$$

and so $f_n(\mathbf{x})\,\mathrm{co}_n^R\,f_n(\mathbf{y})$. □

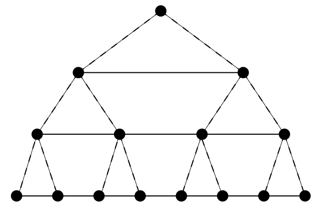


Figure 5.5: The first levels of the approximation sequence $(\mathcal{R}_n, f_n)_{n\in\omega}$

Fig. 5.5 shows the first four tolerance spaces of the approximation sequence: the dashed lines give the maps f_n.

By the same argument used for negative-digit representations, the inverse limit of the approximation sequence $(\mathcal{R}_n, f_n)$ is (isomorphic to) R_∞, with projections $\pi_i : R_\infty \to R_i$ defined by letting $\pi_i(\mathbf{x})$ be the prefix of length i of the sequence $\mathbf{x}$. Instead of using the notion of resolution time, we directly prove that the sequence has transitive limit by first showing that:

Lemma 5.7.2 *For any two distinct sequences $\mathbf{x}, \mathbf{y} \in R_\infty$, the following are equivalent:*

1. *there is some finite sequence $\mathbf{z}$ such that $\mathbf{x} = \mathbf{z}1\mathbf{0}$ and $\mathbf{y} = \mathbf{z}0\mathbf{1}$ (or viceversa), where $\mathbf{0}$ and $\mathbf{1}$ denote the infinite sequence of 0's and 1's, respectively;*
2. *the relation $\mathbf{x}\,\mathrm{co}^R_\infty\,\mathbf{y}$ holds;*
3. *we have $\varphi(\mathbf{x}) = \varphi(\mathbf{y})$.*

Proof: (1) $\Longrightarrow$ (2). Let $k = |\mathbf{z}|$. We shall prove that, for all i, $\pi_i(\mathbf{x})\mathrm{co}^R_i\pi_i(\mathbf{y})$, considering two cases. If $i \le k$, then $\pi_i(\mathbf{x})$ and $\pi_i(\mathbf{y})$ coincide (they are both the prefix of length i of the sequence $\mathbf{z}$). If $i > k$, then

$$
\begin{aligned}
|\varphi(\pi_i(\mathbf{x})) - \varphi(\pi_i(\mathbf{y}))| &= \left|\varphi(\mathbf{z}) + \frac{1}{2^{k+1}} - \varphi(\mathbf{z}) - \sum_{n=k+2}^{i} \frac{1}{2^n}\right| = \\
&= \left|\frac{1}{2^{k+1}} - \frac{1}{2^{k+2}} \sum_{n=0}^{i-(k+2)} \frac{1}{2^i}\right| = \left|\frac{1}{2^{k+1}}(1 - 1 + \frac{1}{2^{i-(k+2)+1}})\right| = \frac{1}{2^i}
\end{aligned}
$$

so $\pi_i(\mathbf{x})\ \mathrm{co}^R_i\ \pi_i(\mathbf{y})$, and thus $\mathbf{x}\,\mathrm{co}^R_\infty\,\mathbf{y}$;
(2) $\Longrightarrow$ (3). Just observe that $|\varphi(\mathbf{x}) - \varphi(\mathbf{y})|$ is at most equal, for every i, to $|\varphi(\pi_i(\mathbf{x})) - \varphi(\pi_i(\mathbf{y}))| + \sum_{n=i+1}^{\infty} 2^{-n}$ which is at most (using the hypothesis) $2/2^i = 1/2^{i-1}$. Since this is true for every i, we obtain the result.
(3) $\Longrightarrow$ (1). Let $\mathbf{z}$ be the longest common prefix of $\mathbf{x}$ and $\mathbf{y}$ (this must be finite, because $\mathbf{x} \neq \mathbf{y}$). Suppose w.l.o.g. that $\mathbf{x} = \mathbf{z}1\mathbf{v}$ and $\mathbf{y} = \mathbf{z}0\mathbf{w}$. Now, if $k = |\mathbf{z}|$, we have:

$$
\begin{aligned}
\varphi(\mathbf{x}) &= \varphi(\mathbf{z}) + \frac{1}{2^{k+1}} + \frac{\varphi(\mathbf{v})}{2^{k+1}} \\
\varphi(\mathbf{y}) &= \varphi(\mathbf{z}) + \frac{\varphi(\mathbf{w})}{2^{k+1}}.
\end{aligned}
$$

Since $\varphi(\mathbf{x}) = \varphi(\mathbf{y})$, we must have $1 + \varphi(\mathbf{v}) = \varphi(\mathbf{w})$. Now, since $\varphi(\mathbf{v}), \varphi(\mathbf{w}) \in [0,1]$, it must be the case that $\varphi(\mathbf{v}) = 0$ and $\varphi(\mathbf{w}) = 1$. The only sequences with values 0 and 1 are $\mathbf{0}$ and $\mathbf{1}$ respectively. □

So, the relation co^R_∞ is transitive, and moreover it coincides with the equivalence relation induced by the fibres[6] of ϕ.

Note that:

Corollary 5.7.1 *Each equivalence class of* co^R_∞ *contains either one or two elements. Moreover, an equivalence class contains two elements if and only if it is mapped (by* φ*) to a dyadic rational different from* 0 *and* 1*.*

Proof: The first part of the statement directly follows from the implication (1) $\iff$ (2) of Lemma 5.7.2. We only have to prove that $\varphi(\mathbf{x})$ is a dyadic rational (different from 0 and 1) if and only if $\mathbf{x}$ has the form $\mathbf{x} = \mathbf{z}\mathbf{1}\mathbf{0}$, which is shown by an easy calculation. □

We are now ready to prove the main theorem of this section, concluding that the inverse image topology induced on $\mathcal{R}_\infty$ is actually (homeomorphic to) the Cantor set. Before doing this, we need a technical lemma:

Lemma 5.7.3 *Let* $\mathcal{B}$ *be the family of subsets of* $\mathbb{R}$ *of the form* (a, b) *where* a, b *are dyadic rationals. Then,* $\mathcal{B}$ *is a base for the standard (Euclidean) topology on* $\mathbb{R}$*.*

Proof: It is clear that $\mathcal{B}$ is a base for a topology Ω. Let $\mathcal{B}'$ be the family of open intervals (which is a base for the standard topology Ω'): we must prove that $\Omega = \Omega'$. Since $\mathcal{B} \subseteq \mathcal{B}'$, we have that Ω' is stronger than Ω (using the observation after Theorem A.3.1). We now prove, using another time Theorem A.3.1, that Ω is stronger than Ω'. Let (a, b) be an interval, and $a < x < b$: since the dyadic rationals are dense in the reals, there will be two dyadic rationals a', b' such that $a < a' < x < b' < b$. So we are done. □

We can now prove the theorem:

Theorem 5.7.1 *Consider the discretely topologized approximation sequence* $(\mathcal{R}_i, f_i)_{i\in\omega}$*. Then, there is a continuous bijection from its limit* $(R^*_\infty, \Omega^*_\infty)$ *to the unit interval (with the usual topology).*

Proof: Consider the function $\varphi : R_\infty \to [0,1]$. This is a surjective map (by Property 5.7.1), and moreover satisfies the assumption (2) of Theorem 5.6.2 (in virtue of the observation made after Lemma 5.7.2). So, in order to use Theorem 5.6.2, we just have to prove that φ is continuous.

Continuity. By Lemma 5.7.3, a subbase for the topology on [0,1] is given by the family of all intervals of the form $[0, \alpha)$ or $(\alpha, 1]$, where $0 < \alpha < 1$ and α is a dyadic number. Using Lemma A.3.1, to prove continuity we just have to show that the counterimage via φ of every interval of the above form is open (in the limit topology). We do this only for intervals of the form $[0, \alpha)$. We already know that α has a representation of the form

[6] Every function $f : A \to B$ induces an equivalence relation $\approx_f$ on the set A, by putting $x \approx_f y$ if and only if $f(x) = f(y)$.

$\mathbf{x} = \mathbf{z01}$, with $k = |z|$ (and $\alpha = \varphi(\mathbf{z}) + \frac{1}{2^{k+1}}$). Now, let $A = \{\mathbf{w} \in R_k : \varphi(\mathbf{w}) < \varphi(\mathbf{z})\}$, and put

$$X = (\bigcup_{\mathbf{w}\in A} \pi_k^{-1}(\mathbf{w})) \cup (\bigcup_{i\in\omega} \pi_{k+i+2}^{-1}(\mathbf{z}01^i0))$$

where 1^i denotes a sequence of i 1's. This is an open subset of R_∞ (it is a union of the elements of the subbase, as explained in Lemma 5.5.1). We shall prove that $\varphi^{-1}([0,\alpha)) = X$, thus proving continuity.
Let $\mathbf{y} \in X$; we have two cases. If $\mathbf{y} = \mathbf{wv}$, where $\mathbf{w} \in A$, then

$$\varphi(\mathbf{y}) = \varphi(\mathbf{w}) + \frac{\varphi(\mathbf{v})}{2^k} \leq \varphi(\mathbf{w}) + \frac{1}{2^k}$$

and, since $\mathbf{w} \in A$, $\varphi(\mathbf{w}) < \varphi(\mathbf{z})$, i.e., $\varphi(\mathbf{w}) \leq \varphi(\mathbf{z}) - \frac{1}{2^k}$; so $\varphi(\mathbf{y}) \leq \varphi(\mathbf{z}) < \alpha$. Otherwise, we have $\mathbf{y} = \mathbf{z}01^i0\mathbf{v}$, whose value is

$$\varphi(\mathbf{y}) < \varphi(\mathbf{z}) + \frac{1}{2^{k+1}}$$

because $1^i0\mathbf{v}$ is a sequence which *must* contain a 0 somewhere.
For the converse, suppose that $\beta \in [0,\alpha)$, and let $\mathbf{y}$ be a representation of β (i.e., $\varphi(\mathbf{y}) = \beta$). We will show that $\mathbf{y} \in X$. In fact, consider the decomposition of $\mathbf{y}$ as $\mathbf{y} = \pi_k(\mathbf{y})\mathbf{v}$; we have three cases:

- if $\varphi(\pi_k(\mathbf{y})) < \varphi(\mathbf{z})$ then $\pi_k(\mathbf{y}) \in A$ and so $\mathbf{y} \in X$;
- if $\varphi(\pi_k(\mathbf{y})) = \varphi(\mathbf{z})$ then $\pi_k(\mathbf{y}) = \mathbf{z}$ (because the function φ is injective when restricted to each finite level); moreover, $\mathbf{v}$ cannot start with a 1, because otherwise the value of $\mathbf{y}$ would be at least $\varphi(\mathbf{z}) + \frac{1}{2^{k+1}} = \alpha$. Finally, $\mathbf{v}$ cannot have the form $\mathbf{v} = \mathbf{01}$, because we would have the same problem as before; so, necessarily, $\mathbf{y} \in X$;
- the case $\varphi(\pi_k(\mathbf{y})) > \varphi(\mathbf{z})$ is impossible, because this would mean $\varphi(\pi_k(\mathbf{y})) - \frac{1}{2^k} \geq \varphi(\mathbf{z})$; so we would have $\beta = \varphi(\pi_k(\mathbf{y})) + \frac{\varphi(\mathbf{v})}{2^k} \geq \varphi(\mathbf{z}) + \frac{1}{2^k} + \frac{\varphi(\mathbf{v})}{2^k}$, which is in turn equal to $\alpha - \frac{1}{2^{k+1}} + \frac{1+\varphi(\mathbf{v})}{2^k}$. But now $\beta < \alpha$ implies $\alpha - \frac{1}{2^{k+1}} + \frac{1+\varphi(\mathbf{v})}{2^k} < \alpha$, i.e., $\varphi(\mathbf{v}) < -\frac{1}{2}$: a contradiction! $\square$

Note that in this case we have *no homeomorphism*, though, because the map φ is not an open map. As a matter of fact, we shall prove in a while that the limit space we have built is actually homeomorphic to the (standard) Cantor set.

The *Cantor set* is the set C of all those real numbers which can be written as $\sum_{i=1}^{\infty} \frac{\epsilon_i}{3^i}$, where $\epsilon_i \in \{0,2\}$ for all $i > 0$; since $C \subseteq [0,1]$, we can endow it with the subspace topology induced by the Euclidean topology on $\mathbb{R}$.

Theorem 5.7.2 *The space $(R_\infty, \Omega_\infty)$ is homeomorphic to the Cantor set.*

Proof: Define a map $\rho : R_\omega \to C$ by putting

$$\rho(\mathbf{x}) = \sum_{i=1}^{|\mathbf{x}|} \frac{2x_i}{3^i}.$$

The function is clearly well-defined (i.e., for all $\mathbf{x}$ we have $\rho(\mathbf{x}) \in C$) and its restriction to R_∞ is surjective. The function is also injective on the same set; in fact, suppose that $\rho(\mathbf{x}) = \rho(\mathbf{y})$, i.e., $\sum_{i=1}^{\infty} \frac{x_i}{3^i} = \sum_{i=1}^{\infty} \frac{y_i}{3^i}$. Working by induction on i, we prove that $x_i = y_i$:

- suppose, by contradiction, that $x_i = 0$ and $y_i = 1$ (the other case is proved analogously): then $\sum_{i=1}^{\infty} \frac{x_i}{3^i} \leq \sum_{i=2}^{\infty} \frac{1}{3^i}$ which equals $\frac{1}{9}\frac{3}{2} = \frac{1}{6}$. Conversely $\sum_{i=1}^{\infty} \frac{y_i}{3^i} \geq \frac{1}{3}$, which contradicts $\rho(\mathbf{x}) = \rho(\mathbf{y})$;

- the inductive step is proved much in the same way.

The proof that ρ is continuous is similar to the one we gave in Theorem 5.7.1 for φ, and thus omitted. We prove that $\rho^{-1} : C \to R_\infty$ is continuous; suppose that $O \subseteq C$ is open (i.e., for all $x \in O$ there exists $\epsilon_x > 0$ such that, for every $y \in C$, if $|x-y| < \epsilon_x$ then $y \in O$). We prove that $\rho^{-1}(O)$ is open. Let $\mathbf{x} \in \rho^{-1}(O)$, and take n as large as $3^{-n} < \epsilon_{\rho(\mathbf{x})}$. If $\mathbf{y} \in \pi_n^{-1}(\pi_n(\mathbf{x}))$ then $\mathbf{y} = \pi_n(\mathbf{x})\mathbf{y}'$ and $\mathbf{x} = \pi_n(\mathbf{x})\mathbf{x}'$. So, we have:

$$|\rho(\mathbf{x}) - \rho(\mathbf{y})| = \frac{|\rho(\mathbf{x}') - \rho(\mathbf{y}')|}{3^n} < \epsilon_{\rho(\mathbf{x})}|\rho(\mathbf{x}') - \rho(\mathbf{y}')| < \epsilon_{\rho(\mathbf{x})}$$

and so, since O is open, $\rho(\mathbf{y}) \in O$, i.e., $\mathbf{y} \in \rho^{-1}(O)$. Since this is true for every $\mathbf{y}$, we obtain that

$$\forall \mathbf{x} \in \rho^{-1}(O) \exists n \in \omega.\ \pi_n^{-1}(\pi_n(\mathbf{x}))$$

which implies that $\rho^{-1}(O)$ is open. □

Notice that, as a consequence, we obtain the following standard result, which is proved in a different way in Gaal [Gaa64] (Lemma IV.9.3):

Corollary 5.7.2 *The unit interval is a continuous image of the Cantor set.*

Proof: This just combines the results of Theorems 5.7.1 and 5.7.2. In fact, let ρ be the homeomorphism built in Theorem 5.7.2, h be the continuous bijection of Theorem 5.7.1, and ψ be the natural injection of $\mathcal{T}$ into $\mathcal{T}^*$ (which is continuous, by definition of reduced space). Then, the function $h \circ \psi \circ \rho^{-1}$ is continuous (because it is a composition of continuous maps) and surjective (because all the functions are such). Note that ψ is not injective, and thus the composition will not be a bijection. □

5.8 Approximating the unit interval

In this section, we shall return to the negative digit representation for the real numbers we discussed in Section 5.4, and study in detail the structure of the limit space, showing that the topology induced in the usual way is actually homeomorphic to the unit interval. For sake of simplicity, we shall limit ourselves to the case of a 2-NDR, and thus put $\Sigma = \{\overline{1}, 0, 1\}$, where $\overline{1}$ abbreviates -1. The function ϕ, and the tolerance spaces $\mathcal{R}_n = (T_n, \mathrm{co}_n)$ are defined as in Section 5.4, taking $b = 2$.

This time, the inverse limit $\mathcal{R}_\infty = (T_\infty, \mathrm{co}_\infty)$ is defined by taking $T_\infty = \Sigma^\omega$, with projections $\pi_k : \mathcal{R}_\infty \to \mathcal{R}_k$ defined by

$$\pi_k(\mathbf{x}) = x_1 x_2 \cdots x_k$$

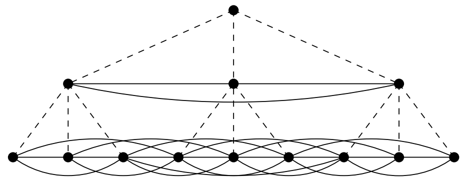


Figure 5.6: The approximation sequence $(\mathcal{R}_n, f_n)$ $(n = 0, 1, 2)$

and limit tolerance relation given by

$$\mathbf{x} \operatorname{co}_\infty \mathbf{y} \text{ iff } \forall k \in \omega . \pi_k(\mathbf{x}) \operatorname{co}_k \pi_k(\mathbf{y}).$$

We have already proved (Theorem 5.4.2) that co_∞ is an equivalence relation, by using the technique of intransitive triples. Now, we shall re-obtain the same result in a direct fashion, by proving that co_∞ is indeed the equivalence relation induced by the fibres of ϕ.

Lemma 5.8.1 *For all $\mathbf{x}, \mathbf{y} \in T_\infty$, we have that $\mathbf{x} \operatorname{co}_\infty \mathbf{y}$ if and only if $\phi(\mathbf{x}) = \phi(\mathbf{y})$.*

Proof: Suppose that $\mathbf{x} \operatorname{co}_\infty \mathbf{y}$; then, for all $k \in \omega$ we have $\pi_k(\mathbf{x}) \operatorname{co}_k \pi_k(\mathbf{y})$. Thus $|\phi(\pi_k(\mathbf{x})) - \phi(\pi_k(\mathbf{y}))| \leq 2b^{-k}$. Hence, if we decompose $\mathbf{x} = \pi_k(\mathbf{x})\mathbf{v}$ and $\mathbf{y} = \pi_k(\mathbf{y})\mathbf{w}$, we obtain

$$\begin{aligned} |\phi(\mathbf{x}) - \phi(\mathbf{y})| &= |\phi(\pi_k(\mathbf{x})) + \phi(\mathbf{v})b^{-k} - \phi(\pi_k(\mathbf{y})) - \phi(\mathbf{w})b^{-k}| \leq \\ &\leq |\phi(\pi_k(\mathbf{x})) - \phi(\pi_k(\mathbf{y}))| + |\phi(\mathbf{v})b^{-k} - \phi(\mathbf{w})b^{-k}| \leq 2b^{-k} + 2b^{-k} = 4b^{-k} \end{aligned}$$

and this is true for all k, so $\phi(\mathbf{x}) = \phi(\mathbf{y})$.
For the converse, suppose that $\phi(\mathbf{x}) = \phi(\mathbf{y})$. Then, using the same decomposition for $\mathbf{x}$ and $\mathbf{y}$ as before, we obtain that, for all $k \in \omega$:

$$\begin{aligned} |\phi(\pi_k(\mathbf{x})) - \phi(\pi_k(\mathbf{y}))| &= |\phi(\mathbf{x}) - \phi(\mathbf{v})b^{-k} - \phi(\mathbf{y}) + \phi(\mathbf{w})b^{-k}| = \\ &= b^{-k}|\phi(\mathbf{w}) - \phi(\mathbf{v})| \leq 2b^{-k} \end{aligned}$$

and thus $\pi_k(\mathbf{x}) \operatorname{co}_k \pi_k(\mathbf{y})$ holds for all $k \in \omega$. □

So, ϕ establishes a one-to-one correspondence between the equivalence classes of $\mathcal{R}_\infty$ (i.e., the elements of $\mathcal{R}_\infty^*$) and the points of $[-1, 1]$. Notice that, in general, each equivalence class may contain infinitely many point (differently from the case of positive digit representation, where each class contains at most two elements, as we showed in Corollary 5.7.1). The reason is that, in this case, the structure of each level is much more complex than before, owing to the fact that the intersection between intervals may be non-trivial (of non-zero measure), and moreover the same interval may have more than one representation on each level. For example, the two sequences $\overline{1}1$ and $0\overline{1}$ are both mapped by ϕ to the same real number $-\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$, and so the corresponding points are equivalent in the space $\mathcal{R}_2$. Said otherwise, the tolerance spaces $\mathcal{R}_n$ are *not* reduced. In Fig. 5.6, we present the first three levels of the approximation sequence, with the same conventions as before (the dashed lines represent the truncation functions f_i).

As it is easy to see, the tolerance spaces in the approximation sequence tend to have a structure which gets exponentially more and more complicated; in order to have a better

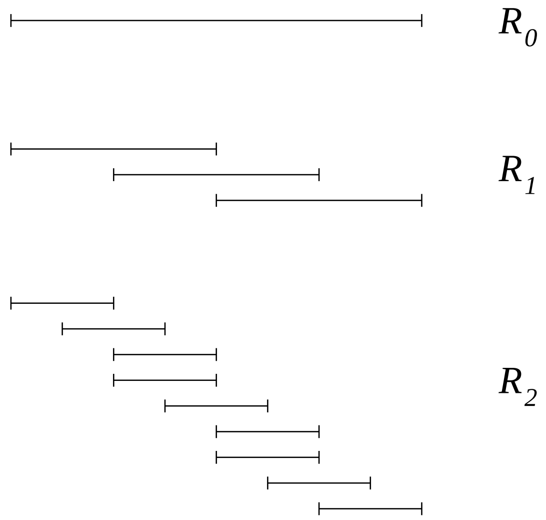


Figure 5.7: An "interval" representation of the approximation sequence

understanding of what is really going on, we present in Fig. 5.7 the "interval" version of $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_2$. For each tolerance space, we have represented each sequence $\mathbf{x} \in T_i$ ($i = 0, 1, 2$) by the interval it represents $[\phi(\mathbf{x}) - b^{-i}, \phi(\mathbf{x}) + b^{-i}]$. Interval overlapping gives rise to the presence of an edge in the tolerance space.

We are now ready to prove the main theorem of this section:

Theorem 5.8.1 *Consider the approximation sequence $(\mathcal{R}_n, f_n)_{n\in\omega}$, with each space discretely topologized. Then, the reduced limit $(\mathcal{R}^*_\infty, \Omega^*_\infty)$ is homeomorphic to the unit interval* $[0, 1]$.

Proof: We shall make use of Theorem 5.6.2, proving that $\phi : T_\infty \to [-1, 1]$ is a continuous open surjective map satisfying the constraint (2) of Theorem 5.6.2. The map ϕ is surjective, by Lemma 5.4.1, and satisfies the constraint, by Lemma 5.8.1. So, we just need to show that it is continuous and open.

Continuity. We already know that a subbase of $[-1, 1]$ is given by the family of intervals of the form $[-1, \alpha)$ or $(\alpha, 1]$, with $-1 < \alpha < 1$ dyadic (see Lemma 5.7.3). So, in order to prove that ϕ is continuous, using Lemma A.3.1, we just have to check that the counter-image of each of these intervals is open: we do this only for the case $[-1, \alpha)$. Since α is a dyadic rational, it will have a representation[7] of the form $\mathbf{x} = \mathbf{z01}$, with $k = |\mathbf{z}|$ (and $\alpha = \phi(\mathbf{z}) + \frac{1}{2^{k+1}}$). Now, let

$$A = \{\mathbf{w} \in T_k : \phi(\mathbf{w}) < \phi(\mathbf{z})\}$$

and, for each $i \in \omega$,

$$B_i = \{\mathbf{w} \in T_{k+i+2} : \phi(\mathbf{w}) \leq \phi(\mathbf{z}01^i0)\}.$$

Define

$$X = \left(\cup_{\mathbf{w}\in A}\pi_k^{-1}(\mathbf{w})\right) \cup \left(\cup_{i\in\omega} \cup_{\mathbf{w}\in B_i} \pi_{k+i+2}^{-1}(\mathbf{w})\right),$$

[7] To be more precise, this happens unless $\alpha = 0$; but we can easily get rid of this case, by observing that the intervals $[-1, 0)$ and $(0, 1]$ can be obtained as $\cup_{k>0}[-1, -2^{-k})$ and $\cup_{k>0}(2^{-k}, 1]$, respectively.

which is an open subset of T_∞, because it is a union of elements of the subbase $\mathcal{S}_\infty$ (see Property 5.5.1). We shall prove that $\phi^{-1}([-1,\alpha)) = X$.
For the right-to-left inclusion, let $\mathbf{y} \in X$; we have two cases:

1. If $\mathbf{y} = \mathbf{wv}$ with $\mathbf{w} \in A$, then $\phi(\mathbf{y}) = \phi(\mathbf{w}) + \frac{\phi(\mathbf{v})}{2^k} \le \phi(\mathbf{w}) + \frac{1}{2^k}$; but $\phi(\mathbf{w}) < \phi(\mathbf{z})$, i.e., $\phi(\mathbf{w}) \le \phi(\mathbf{z}) - \frac{1}{2^k}$; so $\phi(\mathbf{y}) \le \phi(\mathbf{z}) < \alpha$, and thus finally $\phi(\mathbf{y}) \in [-1,\alpha)$, which implies $\mathbf{y} \in \phi^{-1}([-1,\alpha))$.

2. Suppose there exists an $i \in \omega$ such that $\mathbf{y} = \mathbf{wv}$ for some $\mathbf{w} \in B_i$. Then, we have:

$$\begin{aligned}
\phi(\mathbf{y}) &= \phi(\mathbf{w}) + \frac{\phi(\mathbf{v})}{2^{k+i+2}} \le \phi(\mathbf{z}01^i0) + \frac{\phi(\mathbf{v})}{2^{k+i+2}} = \\
&= \phi(\mathbf{z}) + \sum_{j=k+2}^{k+i+1} \frac{1}{2^j} + \frac{\phi(\mathbf{v})}{2^{k+i+2}} = \phi(\mathbf{z}) + \frac{1}{2^{k+2}} \sum_{j=0}^{i-1} \frac{1}{2^j} + \frac{\phi(\mathbf{v})}{2^{k+i+2}} = \\
&= \phi(\mathbf{z}) + \frac{1}{2^{k+2}} \frac{1-\frac{1}{2^i}}{1-\frac{1}{2}} + \frac{\phi(\mathbf{v})}{2^{k+i+2}} = \phi(\mathbf{z}) + \frac{1}{2^{k+1}} \frac{2^i-1}{2^i} + \frac{\phi(\mathbf{v})}{2^{k+i+2}} \le \\
&= \phi(\mathbf{z}) + \frac{2^i-1}{2^{k+i+1}} + \frac{1}{2^{k+i+2}} = \phi(\mathbf{z}) + \frac{2^{i+1}-1}{2^{k+i+2}} = \alpha - \frac{1}{2^{k+1}} + \frac{2^{i+1}-1}{2^{k+i+2}} = \\
&= \alpha + \frac{2^{i+1}-2^{i+1}-1}{2^{k+i+2}} = \alpha - \frac{1}{2^{k+i+2}} < \alpha
\end{aligned}$$

so $\phi(\mathbf{y}) \in [-1,\alpha)$, i.e., $\mathbf{y} \in \phi^{-1}([-1,\alpha))$.

For the left-to-right inclusion, suppose that $\phi(\mathbf{y}) = \beta \in [-1,\alpha)$. We prove that $\mathbf{y} \in X$. Decompose $\mathbf{y}$ as $\mathbf{y} = \pi_k(\mathbf{y})\mathbf{v}$; we have two cases:

1. If $\phi(\pi_k(\mathbf{y})) < \phi(\mathbf{z})$: then $\pi_k(\mathbf{y}) \in A$ and so $\mathbf{y} \in X$.

2. If $\phi(\pi_k(\mathbf{y})) \ge \phi(\mathbf{z})$, suppose, by contradiction, that there is no prefix of $\mathbf{y}$ (longer than $k+1$) belonging to $\cup_{i\in\omega} B_i$. This would mean that, for all $i \in \omega$,

$$\begin{aligned}
\phi(\pi_{k+i+2}(\mathbf{y})) > \phi(\mathbf{z}01^i0) &= \phi(\mathbf{z}) + \frac{1}{2^{k+2}} \sum_{j=0}^{i-1} \frac{1}{2^j} = \\
&= \alpha - \frac{1}{2^{k+1}} + \frac{2^i-1}{2^{k+i+1}} = \alpha + \frac{2^i-2^i-1}{2^{k+i+1}} = \alpha - \frac{1}{2^{k+i+1}}.
\end{aligned}$$

But $\phi(\mathbf{y}) \ge \phi(\pi_{k+i+2}(\mathbf{y})) - \frac{1}{2^{k+i+2}}$ and thus, using the previous inequality,

$$\phi(\mathbf{y}) > \alpha - \frac{1}{2^{k+i+1}} - \frac{1}{2^{k+i+2}} = \alpha - \frac{3}{2^{k+i+2}}$$

for all $i \in \omega$. This means that $\beta = \phi(\mathbf{y}) \ge \alpha$, contradicting the hypothesis.

Open map. In order to prove that ϕ is open, let $X \subseteq T_\infty$ be an open set, and $\alpha \in \phi(X)$. This means that there is some $\mathbf{x} \in X$ such that $\alpha = \phi(\mathbf{x})$. Since X is open, there must be some $k \in \omega$ and $\mathbf{v} \in T_k$ such that $\mathbf{x} \in \pi_k^{-1}(\mathbf{v}) \subseteq X$. But $\alpha = \phi(\mathbf{x}) \in \phi(\pi_k^{-1}(\mathbf{v})) = [\phi(\mathbf{v})-2^{-k}, \phi(\mathbf{v})+2^{-k}] \subseteq \phi(X)$, so there is a whole open ball including α entirely contained in $\phi(X)$. Thus $\phi(X)$ is an open subset of $[-1,1]$. □

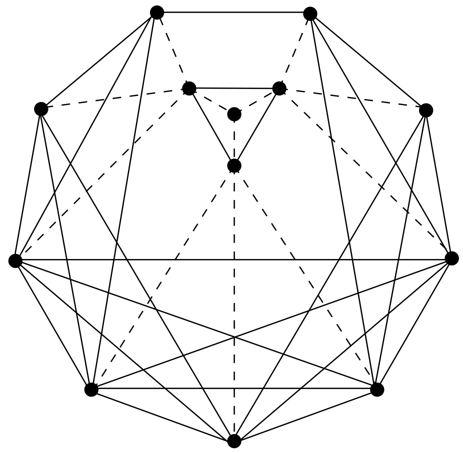


Figure 5.8: The approximation sequence $(\mathcal{R}'_n, f_n)$ $(n = 0, 1, 2)$

5.9 Approximating the circle

In this section, we shall provide an approximation sequence for the circle, by slightly modifying the one previously given for the unit interval. This is a way to suggest how complex (topological) spaces may be described by a sequence of finite approximations, with discrete topologies.

Let $\mathcal{R}_n = (T_n, \mathrm{co}_n)$ be the approximation sequence of 2-NDR, as considered in the previous section. We define a new sequence $\mathcal{R}'_n = (T_n, \mathrm{co}'_n)$, by letting

$$\mathrm{co}'_n = \mathrm{co}_n \cup \{(\overline{1}^n, 1^n), (1^n, \overline{1}^n)\}.$$

In practice, the tolerance space $\mathcal{R}'_n$ is the same as $\mathcal{R}_n$, but has just one additional edge connecting the two extreme points corresponding to a sequence of 1's and a sequence of $\overline{1}$'s. Fig. 5.8 shows the first three levels of this approximation sequence: the diagram is obtained by "folding" the one presented in Fig. 5.6.
Now, we would like to characterize precisely the limit tolerance relation for the space $\mathcal{R}'_\infty$. This is done in the following

Lemma 5.9.1 *Let $\mathbf{x}, \mathbf{y} \in T_\infty$ be two infinite sequences. Then, $\mathbf{x}\, \mathrm{co}'_\infty\, \mathbf{y}$ holds if and only if either $\phi(\mathbf{x}) = \phi(\mathbf{y})$ or $\{\phi(\mathbf{x}), \phi(\mathbf{y})\} = \{-1, 1\}$. Thus, in particular, the sequence $(\mathcal{R}'_n, f_n)$ is globally transitive.*

Proof: The "if" part is a straightforward consequence of Lemma 5.8.1 and of the way we defined co'_n. For the other direction, suppose $\mathbf{x}\, \mathrm{co}'_\infty\, \mathbf{y}$; this means that, for all $k \in \omega$, $\pi_k(\mathbf{x})\, \mathrm{co}'_k\, \pi_k(\mathbf{y})$. Let $\mathrm{co}''_k = \mathrm{co}'_k \setminus \mathrm{co}_k = \{(\overline{1}^k, 1^k), (1^k, \overline{1}^k)\}$; note that $\pi_k(\mathbf{x})\, \mathrm{co}''_k\, \pi_k(\mathbf{y})$ implies that, for all $i \leq k$, also $\pi_i(\mathbf{x})\, \mathrm{co}''_k\, \pi_k(\mathbf{y})$. So, we can put

$$I = \{k : \pi_k(\mathbf{x})\, \mathrm{co}''_k\, \pi_k(\mathbf{y})\}$$

which is, for the above remark, a downward-closed set of indices. If I is infinite, then clearly $\{\mathbf{x}, \mathbf{y}\} = \{\mathbf{1}, \overline{\mathbf{1}}\}$ and thus $\{\phi(\mathbf{x}), \phi(\mathbf{y})\} = \{-1, 1\}$; if I is empty, then necessarily $\pi_k(\mathbf{x})\, \mathrm{co}_k\, \pi_k(\mathbf{y})$ holds for all k, and so $\phi(\mathbf{x}) = \phi(\mathbf{y})$ (by Lemma 5.8.1).

Now, suppose that I is finite and non-empty, and let $k = \max I$. Without loss of generality, we can suppose that $\mathbf{x} = \overline{1}^k\mathbf{v}$ and $\mathbf{y} = 1^k\mathbf{w}$, where $\mathbf{v}$ does not start with a $\overline{1}$, or $\mathbf{w}$ does not start with a 1. Since $\pi_i(\mathbf{x})\ \mathrm{co}'_i\ \pi_i(\mathbf{y})$ holds also for $i > k$, necessarily we must have $\pi_i(\mathbf{x})\ \mathrm{co}_i\ \pi_i(\mathbf{y})$ for every $i > k$; i.e., for all $i > k$, we have:

$$|\phi(\pi_i(\mathbf{x})) - \phi(\pi_i(\mathbf{y}))| \leq \frac{1}{2^{i-1}}. \tag{5.1}$$

But:

$$\phi(\pi_i(\mathbf{x})) = \phi(\overline{1}^k) + \frac{\phi(\pi_{k-i}(\mathbf{v}))}{2^k}$$

and

$$\phi(\pi_i(\mathbf{y})) = \phi(1^k) + \frac{\phi(\pi_{k-i}(\mathbf{w}))}{2^k}.$$

Now, observe that

$$\phi(\overline{1}^k) = -\sum_{i=1}^{k}\frac{1}{2^i} = -\frac{1}{2}\sum_{i=0}^{k-1}\frac{1}{2^i} = \frac{1}{2}\frac{\frac{1}{2^k}-1}{1-\frac{1}{2}} = \frac{1-2^k}{2^k}.$$

and so, for every $i > k$

$$\begin{aligned}\phi(\pi_i(\mathbf{x})) - \phi(\pi_i(\mathbf{y})) &= \frac{1-2^k}{2^k} + \frac{\phi(\pi_{k-i}(\mathbf{v}))}{2^k} - \frac{2^k-1}{2^k} - \frac{\phi(\pi_{k-i}(\mathbf{w}))}{2^k} = \\ &= \frac{2-2^{k+1}}{2^k} + \frac{\phi(\pi_{k-i}(\mathbf{v})) - \phi(\pi_{k-i}(\mathbf{w}))}{2^k}.\end{aligned}$$

In order to make (5.1) hold, we must have:

$$\left|\frac{\phi(\pi_{k-i}(\mathbf{v})) - \phi(\pi_{k-i}(\mathbf{w})) + 2}{2^k} - 2\right| \leq \frac{1}{2^{i-1}}. \tag{5.2}$$

But $\phi(\pi_{k-i}(\mathbf{v})) - \phi(\pi_{k-i}(\mathbf{w})) \in [-2, 2]$, and so

$$\frac{\phi(\pi_{k-i}(\mathbf{v})) - \phi(\pi_{k-i}(\mathbf{w})) + 2}{2^k} - 2 \in [-2, \frac{4}{2^k} - 2];$$

thus, its absolute value belongs to $[2 - \frac{1}{2^{k-2}}, 2]$. For the inequality (5.2) to hold, it is necessary that

$$\begin{aligned} 2 - \frac{1}{2^{k-2}} &\leq \frac{1}{2^{i-1}} \\ 8 - \frac{1}{2^k} &\leq \frac{2}{2^i} \\ 8 &\leq \frac{2 + 2^{i-k}}{2^i}. \end{aligned}$$

This should hold for every $i > k$, and k is fixed. So, in particular

$$8 \leq \lim_{i\to\infty}\frac{2+2^{i-k}}{2^i} = 2^{-k}$$

which is impossible. □

So, the limit space $\mathcal{R}'_\infty$ is a transitive one, with equivalence relation which is the same as co_∞, but which moreover equals the two sequences $\mathbf{1}$ and $\overline{\mathbf{1}}$. Using the result of Theorem 5.8.1, we obtain that

Corollary 5.9.1 *Consider the approximation sequence* $(\mathcal{R}'_n, f_n)_{n\in\omega}$, *with each space discretely topologized. Then, its reduced limit is homeomorphic to the circle*[8] S^1.

Proof: Just remember that S^1 is homeomorphic to $[0,1]$ quotiented with the equivalence which identifies 0 and 1 (see Gaal [Gaa64], Section 14.I). □

[8] The (unit) circle is the subspace of the ordinary (euclidean) space $\mathbb{R}^2$ induced by the set $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Chapter 6

Tolerance spaces and semiorders

6.1 Order and disorder in measurement theory

In the previous chapters, we have been concerned with general tolerance spaces, considering the indifference relation as a "primitive" one, and without investigating the way in which it was generated or produced. In this chapter, we shall restrict our attention to a special but very important class of tolerance spaces, namely, those which are generated by some (particular) kind of partial order[1]. The interest of such spaces stems from the theory of measurement, as better explained in the rest of this section.

Fred S.Roberts, in his encyclopaedic work on Measurement Theory [Rob79], states that *a major difference between a "well-developed" science such as physics and some of the less "well-developed" sciences such as psychology or sociology is the degree to which things are measured*. For this reason, we think that giving firm foundations to the abstract theory of measurement would be of use not only for physics – where well-established methods of measurement are a matter of fact – but also, and especially, for those social sciences which still miss formal tools for doing measurements and reasoning about their results.

When speaking of what measurement is about, Bertrand Russell [Rus37, Chapter XXI] says that:

> Measurement of magnitudes is, in its most general sense, any method by which a unique and reciprocal correspondence is established between all or some of the magnitudes of a kind and all or some of the numbers[2], integral, rational, or real, as the case may be.

Of course, we expect this assignment of numbers to magnitudes to respect certain basic properties of the represented magnitudes, and to preserve them as properties of the numbers used for the representation. So, for example, if a certain body A is "heavier" than B, we expect it to have a greater mass than that of B; and similarly, if A is made of two (disjoint) subparts, say A_1 and A_2, we want the mass of A to equal the sum of the masses of A_1 and A_2.

[1] Part of the results contained here appeared in [Bol96].

[2] Yet, measuring without numbers can also be a sensible and fruitful activity [Rob79]. (The note is by myself)

This way of thinking about measurement could be called "quantitative", as opposed to another approach which we could term "comparative". As Smith [Smi93] asserts, measurement always involves a *comparison* of the measurand with another object of the same kind, and we could consider the result of such comparisons as our measurements.

Whilst the quantitative aspects seem to be the most important in physics, the comparative ones tend to prevail when social sciences are considered. If we take the position that measurement is comparison, we should first decide which kind of comparisons we are going to allow, and what the result of each comparison should be.

Suppose, for example, that we are given a set P of objects and are requested to measure our preferences about the objects in P. We could allow conjoint preference to be considered, but we prefer to restrict ourselves to the case of individual comparisons only. In other words, for any two objects x, y in P we should wonder which is the one we prefer, if any[3]. The result of our comparisons will therefore be a binary relation ρ on P, which we expect to satisfy two axioms for all $x, y, z \in P$:

1. *Irreflexivity:* not $(x \rho x)$, i.e., no element is preferred to itself;
2. *Transitivity:* if $x \rho y$ and $y \rho z$, then $x \rho z$; i.e., if we prefer x to y and y to z then we prefer x to z.

These axioms express a sort of "rationality condition" in the case of preference: a truly rational human being should make judgements that satisfy the axioms, or else he is not acting rationally [Rob79, Rus37]. In the case of physical measurement, these axioms state that our measuring devices "work well"; or, said otherwise, they define the minimal conditions under which measurement can take place.

A relation ρ satisfying the above conditions (1),(2) is called a *strict partial order*[4] (or simply a strict order), and it is usually denoted $<$, while the pair $(P, <)$ is called a (partially) *strictly ordered set* ("poset" for short). When the relation is understood, we simply write P both for the poset and the underlying set; we always assume P to be non-empty.

So, the result of a measurement is a partial order $<$ on the set P of measurands; given two objects x, y in P, they are *comparable* if $x < y$ or $x > y$ (i.e., $y < x$); they are *incomparable* (other terms: indifferent, indistinguishable, independent) if they are not comparable, and in this case we write $x \sim y$. The relation $\sim$ is sometimes called a "disorder", being the complement of (the symmetric closure of) a (strict) order.

What is the correct interpretation for the relation $\sim$ of indifference? This is clearly a tolerance relation, and, in the case of preference, $x \sim y$ simply means that we are indifferent among x and y or, in other words, that x and y can be freely substituted for each other without any gain or loss. Of course, $\sim$ is a reflexive and symmetric relation,

[3] Of course, we could also consider a more complex kind of comparison, where we also have to say "how much the object we prefer is better than the other". This goes outside the scope of this thesis, but see [Rob79] for this enlarged class of comparisons.

[4] In this section, we shall deal mainly with strict partial orders, instead of the kind of partial orders defined in Chapter 3, and will thus omit the adjective "strict". Note that there is a natural one-to-one correspondence between these two kinds of relations, the only difference being represented by (ir)reflexivity. Nevertheless, we prefer to work with strict partial orders both because this is more common in the measurement theory area, and because this fits better our needs.

but it is not transitive in general (when it is transitive, we say that $<$ is a *weak order*; in particular, when $\sim$ coincides with the identity of P, we say that $<$ is a *total*, or linear, order; see also Chapter 3). Classical examples contradicting transitivity of indifference in social sciences have been presented in Section 5.1.

If one wants to model a piece of real world, imprecision of measurement should be respected from the beginning. This approach is well explained by Petri in [Pet80b] (but see also [Pet96]):

> Concerning net theory, we take the standpoint that this imprecision is not only a consequence of our poor abilities to distinguish by direct or indirect observation, but it is in the nature of the measuring process itself and in our relation to the object we measure.

Henceforth, any mathematical technique for modelling processes should take into account the complex nature of the information flows to be represented, and the influence of noise as an extra-structure bearing imprecision.

The previous discussion should have pointed out that tolerance arising from incomparability deserves a special attention; in the current literature, a graph which is the symmetric closure of a (strict) partial order is called a *comparability graph* (see Golumbic [Gol80] for a general survey on this subject). An *incomparability (tolerance) space* is thus the dual[5] of a comparability graph.

Even though we are not interested in the problem of studying the general case of incomparability spaces, we present a very simple characterization of comparability, due to Gouila-Houri [GH62]:

Theorem 6.1.1 (Ghouila-Houri [GH62]) *A graph $G = (V, E)$ is a comparability graph if and only if it does not contain any cycle $v_1, \ldots, v_n$ (with $n > 2$ odd) such that $(v_i, v_{i+2}) \notin E$ for all $i = 1, \ldots, n$ (where addition is modulo n).* $\square$

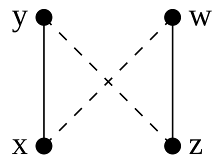
A cycle like that appearing in the statement of the Ghouila-Houri Theorem is called a *forbidden odd-cycle*; thus, an incomparability space is a space whose dual does not contain forbidden odd-cycles.

6.2 Generalities on semiorders and strong noetherianity

In the previous section, we have explained in some detail the rôle of (in)comparability in the theory of tolerance spaces, as applied to measurement of "scalar" values. Now, we are going to introduce some more strict axioms on the nature of the orders arising in measurement, and thus obtain some restrictions on the kind of tolerance spaces generated in this way.

Even though we have taken the approach of considering comparative aspects only, we admit the existence of an ideal function giving the "real" measurements of our objects; i.e.,

[5] The dual of a tolerance space (undirected graph) is the graph having the same set of nodes, and which contains and edge between two nodes exactly when there was no edge in the original space.



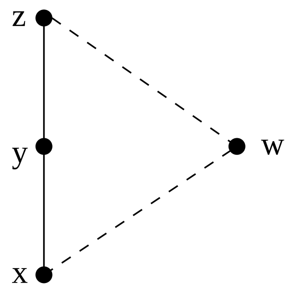


Figure 6.1: Axioms of semiorder

we assume that there exists an (unknown) function $f : P \to \mathbb{R}$ expressing the measures of objects (their masses, their preference rankings etc.). Our order relation simply gives a partial sketch of this function, in the sense that we expect the following relation to hold:

$$x < y \implies f(x) < f(y).$$

This is a sort of soundness assumption of our measurement method: if we conclude that x is lighter than y, then x is *really* lighter than y. Yet, assuming the converse is too much: it would give us a weak order, which is too irrealistic at the light of our previous considerations.

To solve this problem, Luce [Luc56] – motivated by the notion of threshold in psychophysics – suggested that the relation "smaller than" should be substituted by a stronger relation like "sufficiently smaller than"; in other words, he assumed the existence of a threshold value $\delta > 0$ such that:

$$x < y \iff f(x) + \delta < f(y).$$

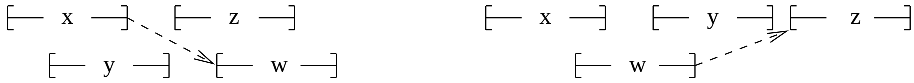
The pair (f, δ) is a *representation* for the poset $(P, <)$. Of course, the value δ is immaterial here: any δ would work equally well, by simply changing the scale f.

It is easy to show that any representable poset P satisfies the following axioms for all $x, y, z, w \in P$:

- *Axiom A1.* if $x < y$ and $z < w$ then $x < w$ or $z < y$;
- *Axiom A2.* if $x < y < z$ then $x \not< w$ or $z \not< w$.

Fig. 6.1 gives a graphical representation of the two axioms, in the form of Hasse diagrams: if the relations indicated by the full lines hold, then one of the dashed relations also holds necessarily.
In Fig. 6.2 we present the axioms by showing an example of how they work in the presence of a representation (here, each point x is represented by a closed interval on the real line, i.e., $[f(x) - \delta, f(x) + \delta]$, and incomparability is determined by interval-overlapping).

A poset satisfying axiom A1 is called an *interval order*; if it also satisifes axiom A2, we say that it is a *semiorder* (see e.g. [Smi93, Fis85, Rob79]). So, representable posets



Axiom (A1) Axiom (A2)

Figure 6.2: Representation of the axioms on the real line

are semiorders; the converse fails in the general case (there are infinite semiorders which have no representations), but holds in the finite case, as shown in the following well-known result:

Theorem 6.2.1 (Scott-Suppes Representation Theorem [SS58]) *If* $(P, <)$ *is a finite semiorder, then there exists a function* $f : P \to \mathbb{R}$ *s.t.*

$$\forall x, y \in P.\, x < y \ \textit{iff}\ f(x) + 1 < f(y).$$

□

In the rest of this chapter, we shall mainly be interested in studying propertiers of a special kind of semiorders (and thus also of the associated tolerance spaces): the strongly noetherian ones.

Consider a poset $(P, <)$; we say that P is *noetherian*[6] if and only if the relation $>$ is noetherian, i.e., iff there does not exist a sequence $x_0, x_1, x_2, \ldots$ with

$$x_0 > x_1 > x_2 > \ldots$$

i.e., the relation $<$ is a well-founded order. We say that P is *strongly noetherian* if and only if, for all $X \subseteq P$:

- if X is a linearly ordered subposet and X has maximum, then X is finite.

It is straightforward to show that a strongly noetherian poset is noetherian as well, but the converse is not true, not even for semiorders; for example, if you consider the ordinal $\omega + 1$, ordered as usual, you will see that it is noetherian but not strongly noetherian (Fig. 6.3).

An informal justification for the use of strongly noetherian semiorders is that in many cases we consider magnitudes which are bounded below in their scales. For example, this is the case of masses, where the scale is bounded by zero (negative masses are not allowed), or also the case of temperatures (the absolute zero is the limit in this case). If we restrict representations to have, say, non-negative values only, we shall surely obtain strongly noetherian semiorders.

We shall now use the Scott-Suppes Theorem for proving a useful property of the descending chains in a semiorder.

[6]The term *conoetherian* may be found instead.

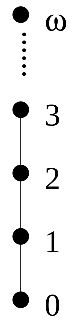


Figure 6.3: The ordinal $\omega + 1$

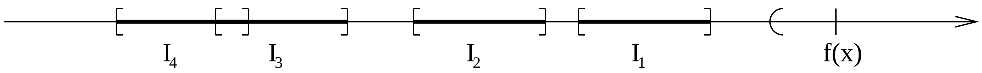


Figure 6.4: Proof of Property 6.2.1

Property 6.2.1 *Let $(P, <)$ be a semiorder, and consider two descending chains of the form:*

$$x = x_0 > x_1 > x_2 > \ldots > x_n$$

$$x = y_0 > y_1 > y_2 > \ldots > y_m$$

with $m > 2n$. There exists an index $i = 1, \ldots, m$ such that

- *either $x_j > y_i > x_{j+1}$ for some $j = 0, \ldots, n-1$*
- *or $x_n > y_i$.*

Proof: Let $X = \{x, x_1, \ldots, x_n, y_1, \ldots, y_m\}$; the restriction of $<$ to X gives rise to a finite semiorder, and so we can use Scott-Suppes Theorem to find a function $f : X \to \mathbb{R}$ representing X over the reals.

For each $k = 1, \ldots, n$ let $I_k = [f(x_k) - 1, f(x_k) + 1]$, and also let $S = (-\infty, f(x) - 1)$ and $T = S \setminus \bigcup_k I_k$. Observe that for each $j = 1, \ldots, m$, $f(y_j) \in S$ (because $y_j < x \implies f(y_j) < f(x) - 1$). If for some j, $f(y_j) \in T$ then either $x_k > y_j > x_{k+1}$ for some k, or $x_n > y_j$ (see Fig. 6.4), as required.

By contradiction, suppose that no j satisfies $f(y_j) \in T$. This means that

$$\forall j = 1, \ldots, m \exists k = 1, \ldots, n.\ f(y_j) \in I_k$$

By the pigeonhole principle, there exists an index k such that I_k contains more than two points: but if an interval of length 2 contains three points, two of them must have distance at most 1, and so they should be indistinguishable; this is a contradiction (no two y_j's are incomparable). $\square$

Let $(P, <)$ be a poset, and define, for each $x \in P$, the set:

$$C(x) = \{x = x_0 > x_1 > \ldots > x_n : n \in \omega\}$$

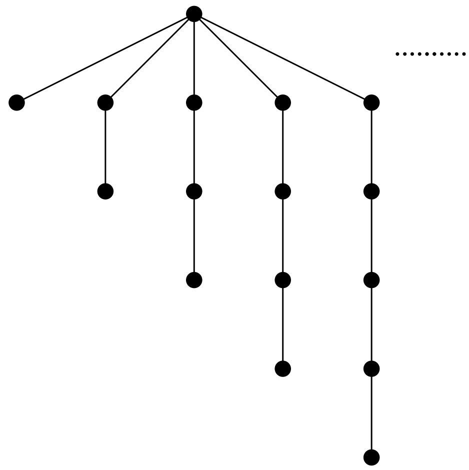


Figure 6.5: An unbounded set of descending chains

(the set of finite descending chains starting from x); let also

$$L(x) = \{|\gamma| - 1 : \gamma \in C(x)\}.$$

Observe that $C(x)$ is closed by chain-prefixes, and $L(x)$ is therefore a downward closed set of natural numbers. So, either $L(x)$ is finite or $L(x) = \omega$. The second case is possible even when P is noetherian, because there might be infinitely many finite descending chains of increasing length starting from a single element (see Fig. 6.5).

This is also possible in the case of a noetherian semiorder (see Fig. 6.3). Yet, we shall prove in a moment that this is impossible in a strongly noetherian semiorder.

Lemma 6.2.1 *Let $(P, <)$ be a poset, and suppose that there exist in P the following descending chains:*

$$\begin{aligned}
\gamma^{(0)} &= x_0^{(0)} \\
\gamma^{(1)} &= x_0^{(1)} > x_1^{(1)} \\
\gamma^{(2)} &= x_0^{(2)} > x_1^{(2)} > x_2^{(2)} \\
&\dots
\end{aligned}$$

where $x_0^{(0)} = x_0^{(1)} = \ldots = x$, and moreover every element occurring in $\gamma^{(k)}$ also occurs somewhere in $\gamma^{(k+1)}$ (for all $k \in \omega$). Then P is not strongly noetherian.

Proof: Just let $X = \{x_i^{(j)} : 0 \leq i \leq j, j \in \omega\}$. Of course, X is an infinite set, and moreover it is linearly ordered. In fact, take $x_i^{(j)}, x_{i'}^{(j')}$ and let $k = \max\{j, j'\}$. Then for some s, s' we have:

$$x_s^{(k)} = x_i^{(j)}, x_{s'}^{(k)} = x_{i'}^{(j')}.$$

But now $x_s^{(k)}$ and $x_{s'}^{(k)}$ are comparable (because they belong to the same chain). So X is an infinite linearly ordered subset of P. But it contains a maximum element x, and so P cannot be strongly noetherian. □

As a consequence, we obtain:

Theorem 6.2.2 *Let $(P, <)$ be a strongly noetherian semiorder. Then, for all $x \in P$, $L(x)$ is a finite set of natural numbers, and we define:*

$$h(x) = \max L(x)$$

(the height[7] of x).

Proof: By contradiction, suppose that $L(x) = \omega$ for some x. Then we find an infinite sequence of longer and longer descending chains starting from x, say:

$$\begin{array}{rcl} \psi^{(0)} & = & x_0^{(0)} \\ \psi^{(1)} & = & x_0^{(1)} > x_1^{(1)} \\ \psi^{(2)} & = & x_0^{(2)} > x_1^{(2)} > x_2^{(2)} \\ \dots & & \end{array}$$

where $x = x_0^{(0)} = x_0^{(1)} = \dots$. Now, we inductively build a new sequence $\gamma^{(k)}$ as follows. We let:

$$\gamma^{(0)} = x_0^{(0)}.$$

Suppose that you have already built $\gamma^{(0)}, \dots, \gamma^{(n)}$, and consider the two chains $\gamma^{(n)}$ and $\psi^{(2n+1)}$. Using Property 6.2.1, we find a new element to "enrich" the chain $\gamma^{(n)}$, and thus we can add it and obtain a longer chain $\gamma^{(n+1)}$. Now the sequence $\gamma^{(0)}, \gamma^{(1)}, \dots$ satisfies the hypothesis of Lemma 6.2.1, and so P cannot be strongly noetherian: a contradiction. □

Here are some easy facts about heights:

Property 6.2.2 *Let $(P, <)$ be a strongly noetherian semiorder. The following hold:*

1. *if $x < y$ then $h(x) < h(y)$;*
2. *if $h(x) = n + 1$ then there exists $y \in P$ with $h(y) = n$ and $y < x$;*
3. *there exists some x such that $h(x) = 0$;*
4. *if $h(x) = h(y)$ then $x \sim y$.*

Proof: (1) Take any descending chain starting from x, say $x = x_0 > x_1 > \dots > x_n$. Then $y > x_0 > x_1 > \dots > x_n$ is a descending chain starting from y having length $n + 1$. This means that:

$$n \in L(x) \implies n + 1 \in L(y)$$

and so $\max L(x) < \max L(y)$.
(2) If $h(x) = n + 1$ then there exists some chain of the form $x = x_0 > x_1 > \dots > x_{n+1}$. Take $y = x_1$. Of course $y < x$, and $h(y) \geq n$ (since $n \in L(y)$). But $y < x$ implies (using (1)) that $h(y) < h(x) = n + 1$, so $h(y) = n$.
(3) Simply use repeatedly (2) and the assumption that $P \neq \emptyset$.
(4) A consequence of (1). □

[7] This use of the word height is somehow non-standard; in the current literature, the height of an element is defined only as far as all the maximal descending chains starting from that element have the same length (the Jordan-Dedekind property).

6.3 Cuts in strongly noetherian semiorders

One way to interpret the indifference relation of a tolerance space is to say that it expresses a sort of "consistency" or "compatibility": two points are in tolerance if and only if they are compatible, i.e., if there is some chance for the real measure to satisfy both constraints (intervals). Clearly, the maximal information one can get is obtained by gathering a maximal set of pairwise compatible measures, i.e., a maximal clique of the tolerance space. A maximal clique has indeed the rôle of a point in the measurement scale we are using.

If we are considering tolerance spaces induced by partial orders, the cliques corresponds to maximal antichains in the order, which we shall from now on call cuts, using the terminology introduced by Petri.

Formally, given a partial order $(P, <)$, a *cut* (*line*) is a maximal set of pairwise incomparable (comparable) elements. In this section, we shall study the structure of cuts in a strongly noetherian semiorder.

From now on, we let $(P, <)$ be any fixed non-empty strongly noetherian semiorder, and use $h(x)$ to denote the height of x. Moreover, for any $X \subseteq P$ we write X^n for

$$X^n = \{x \in X : h(x) = n\}.$$

Throughout the rest of this Chapter, we are assuming that:

Lemma 6.3.1 (Kuratowski's Lemma) *Each set of pairwise (in)comparable elements is included in a line (cut, resp.).*

Kelley [Kel55] observes that this is equivalent to the Axiom of Choice.
Note the following:

Property 6.3.1 *If $x \sim y$ then $|h(x) - h(y)| \leq 1$.*

Proof: By contradiction, suppose that $x \sim y$ with $h(x) = n$ and $h(y) = n + k + 1$ (with $k > 0$). Using Property 6.2.2 (2) we find two elements y', y'' with $y' < y'' < y$ and $h(y') = n$,$h(y'') = n + k$. But $x \sim y'$ (because of Property 6.2.2 (4)) and so $x < y$: a contradiction. □

Now, let $\mathcal{C}_P$ ($\mathcal{L}_P$) be the set of all cuts (lines, resp.) of P. For each cut $C \in \mathcal{C}_P$, define:

$$h(C) = \min\{h(x) : x \in C\} \text{ (the height of } C\text{)}$$

$$\hat{C} = \{x \in C : h(x) = h(C)\} = C^{h(C)} \text{ (the lower part of } C\text{)}.$$

We have that:

Corollary 6.3.1 *For every cut C*

$$\forall x \in C.\, h(C) \leq h(x) \leq h(C) + 1.$$

Proof: Immediate from Property 6.3.1. □

So, if C is a cut of height n, then either all the elements of C have height n, or C contains some elements of height n and some elements of height $n + 1$ (and nothing else). We call a cut C *horizontal* in the first case, i.e., when

$$\forall x \in C.\, h(x) = h(C)$$

and *vertical* otherwise.

Property 6.3.2 *Let C be a cut:*

1. *if C is horizontal, then $\hat{C} = C = P^{h(C)}$;*
2. *if C is vertical, then $\hat{C} \subset C$ and $\hat{C} \subset P^{h(C)}$.*

Proof: (1) The first equality is straightforward. For the second equality, the left-to-right inclusion is obvious. Suppose that $h(x) = h(C)$; then x is incomparable with every element of C and so $x \in C$.
(2) The first inclusion is proper by definition. For the second, suppose by contradiction that $\hat{C} = \{x \in P : h(x) = h(C)\}$, and let $y \in C$ be any element of height $h(C) + 1$. Then we find (Property 6.2.2) an element x with $h(x) = h(C)$ such that $x < y$: which is clearly impossible, since $x, y \in C$. □

A consequence of the previous property is that, for each k, there exists at most one horizontal cut of height k; also observe that two different horizontal cuts are disjoint. Moreover:

Property 6.3.3 *Any two vertical cuts of the same height are non-disjoint.*

Proof: Let C_1, C_2 be any two vertical cuts of height n, and suppose $C_1 \cap C_2 = \emptyset$. Now take any two $x_1 \in C_1$, $x_2 \in C_2$ of height $n + 1$. Since $x_1 \notin C_2$, there must exist some $y_2 \in C_2$ with $y_2 < x_1$, and similarly $\exists y_1 \in C_1.\, y_1 < x_2$. But now either $y_2 < x_2$ or $y_1 < x_1$: both cases lead to a contradiction. □

The following lemma establishes an important characterization of the intersection among vertical cuts of the same height.

Lemma 6.3.2 *Let C_1, C_2 be two (different) vertical cuts of height n. Then, one of the following holds:*

- *$C_1^n \subset C_2^n$ and $C_2^{n+1} \subset C_1^{n+1}$;*
- *$C_2^n \subset C_1^n$ and $C_1^{n+1} \subset C_2^{n+1}$.*

Proof: By Property 6.3.3, $C_1 \cap C_2 \neq \emptyset$. Suppose first that both $C_1^n \setminus C_2^n$ and $C_2^n \setminus C_1^n$ are non-empty, and let $x \in C_1^n \setminus C_2^n$ and $y \in C_2^n \setminus C_1^n$. Since $x \notin C_2$, there exists $x' \in C_2$ such that $x < x'$, and similarly there exists $y' \in C_1$ such that $y < y'$. But these together imply $x < y'$ (which is impossible, since $x, y' \in C_1$) or $y < x'$ (which is also impossible for similar reasons). So we must conclude that:

a) either $C_1^n \subseteq C_2^n$

a') or $C_2^n \subseteq C_1^n$.

With the same kind of argument, we obtain:

b) either $C_1^{n+1} \subseteq C_2^{n+1}$

b') or $C_2^{n+1} \subseteq C_1^{n+1}$.

Observe that a)+b) imply $C_1 \subseteq C_2$, which is clearly impossible (as C_1, C_2 are cuts and are different), and the same is true for a')+b'). So the only other possibilities are those listed in the thesis. □

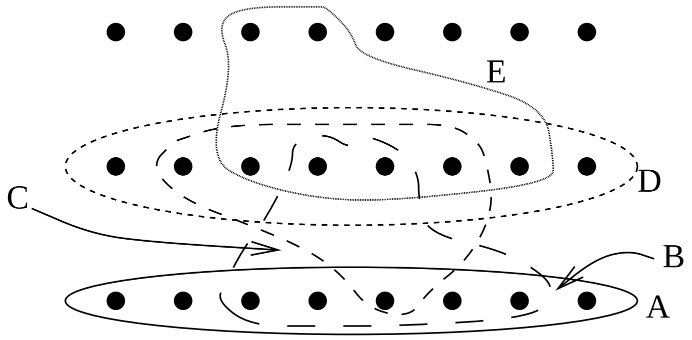


Figure 6.6: The dynamic of cuts in a strongly noetherian semiorder

6.4 A consecutive linear order of cuts

The result of Lemma 6.3.2 is an important step towards our understanding of how the cuts of a strongly noetherian semiorder are structured. To be more precise, we can construct a sort of "dynamics" of cuts, which allows one to pass from one cut to another in a sort of continuous way. Consider the horizontal cut of height 0 in a strongly noetherian semiorder, as sketched in Fig. 6.6 (A). If we want to add some new element of height 1, we must leave out some elements of height 0 (Fig. 6.6 (B)). We go on this way (cut (C)), until no element of height 0 remains, and we reach the horizontal cut of height 1 (D). Then, the process goes on like this starting from (D) and going upwards (E). The movement from one cut to another is unique, as guaranteed by Lemma 6.3.2.

We can formalize this by considering the following relation on cuts:

$$C_1 \prec C_2 \text{ iff } (h(C_1) = h(C_2) \text{ and } \hat{C}_1 \supset \hat{C}_2) \text{ or } h(C_1) < h(C_2).$$

A consequence of Lemma 6.3.2 is that:

Lemma 6.4.1 *The relation $\prec$ is a linear order.*

Proof: It is straightforward to check that $\prec$ is a partial order. Now, take any two cuts C_1, C_2. If $h(C_1) \neq h(C_2)$ then they are somehow $\prec$-related. If otherwise $h(C_1) = h(C_2)$ then, by Lemma 6.3.2 either $\hat{C}_1 \subset \hat{C}_2$ or the converse, and so we are done. □

As an example, consider the semiorder in Fig. 6.7. Here is a list of its cuts (their numbering being arbitrary):

CUT	*TYPE*	*HEIGHT*	*LOWER PART*
$C_1 = \{a, b, c\}$	horizontal	0	$\{a, b, c\}$
$C_2 = \{a, d, e, f\}$	vertical	0	$\{a\}$
$C_3 = \{a, d, e, c\}$	vertical	0	$\{a, c\}$
$C_4 = \{d, f, g\}$	vertical	1	$\{d, f\}$

The ordering here is:

$$C_1 \prec C_3 \prec C_2 \prec C_4.$$

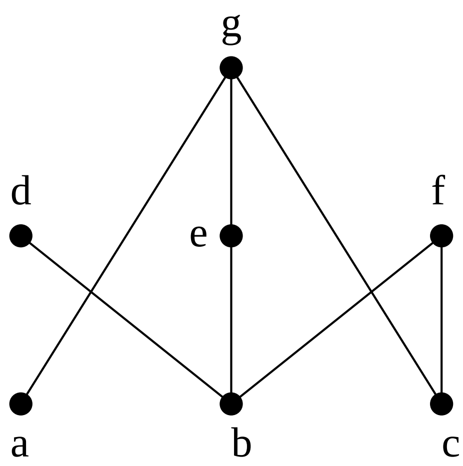


Figure 6.7: A finite semiorder

A linear order $\ll$ on $\mathcal{C}_P$ is *consecutive* if and only if

$$C_1 \ll C_2 \ll C_3 \text{ and } x \in C_1 \cap C_3 \implies x \in C_2,$$

i.e., the set of cuts including a certain element forms an interval in the order $\ll$.

Theorem 6.4.1 *The relation $\prec$ is a consecutive linear order.*

Proof: Suppose that $C_1 \prec C_2 \prec C_3$ with $x \in C_1 \cap C_3$. We distinguish four cases:

1. $h(C_1) = h(C_2) = h(C_3) = n$;
2. $n = h(C_1) < h(C_2) = h(C_3)$;
3. $n = h(C_1) = h(C_2) < h(C_3)$;
4. $h(C_1) < h(C_2) < h(C_3)$.

Case 1. We have $C_3^n \subset C_2^n \subset C_1^n$ and $C_1^{n+1} \subset C_2^{n+1} \subset C_3^{n+1}$. If $h(x) = n$, then $x \in C_3^n$ and so $x \in C_2^n$. If $h(x) = n+1$, then $x \in C_1^{n+1}$ and so $x \in C_2^{n+1}$.
Case 2. In this case $h(x) = n+1$, and $C_3^{n+1} \subset C_2^{n+1}$. But $x \in C_3^{n+1}$ and so $x \in C_2^{n+1}$.
Case 3. In this case also $h(x) = n+1$, $C_2^n \subset C_1^n$ and $C_1^{n+1} \subset C_2^{n+1}$. But $x \in C_1^{n+1}$ and thus $x \in C_2^{n+1}$.
Case 4. This case is impossible, since then $C_1 \cap C_3 = \emptyset$. □

The following table shows the intervals corresponding to the previous example (a cross means that the element belongs to the cut):

Element	C_1	C_3	C_2	C_4
a	x	x	x	
b	x			
c	x	x		
d		x	x	x
e		x	x	
f			x	x
g				x

For all $x \in P$, let $\mathcal{I}_x = \{C \in \mathcal{C}_P : x \in C\}$, and also

$$\mathcal{I}_x^n = \{C \in \mathcal{I}_x, h(C) = n\}.$$

We use the notation $C_1 \preceq C_2$ for $C_1 \prec C_2$ or $C_1 = C_2$. The following Lemma shows that $\mathcal{I}_x$ is in fact a closed interval w.r.t. the relation $\prec$.

Lemma 6.4.2 *For all $x \in P$, there exist $C_x^L, C_x^R \in \mathcal{I}_x$ such that for all $C \in \mathcal{C}_P$ we have:*

$$x \in C \textit{ iff } C_x^L \preceq C \preceq C_x^R.$$

Proof: We only show the existence of C_x^L, and prove the left-to-right implication; the same arguments determine the existence of C_x^R, and the inverse implication is obtained by using simply the consecutivity of $\prec$.
Let x have height n, and first observe that $\mathcal{I}_x$ contains cuts of height $n-1$ and/or n. We consider two cases.
First case. Suppose that no cuts of height $n-1$ include x. Take:

$$C_x^L = (\bigcup_{C \in \mathcal{I}_x} C^n) \cup (\bigcap_{C \in \mathcal{I}_x} C^{n+1}).$$

and observe that:

1. $x \in C_x^L$, because $x \in C^n$ for all $C \in \mathcal{I}_x$.
2. C_x^L is a set of pairwise independent elements; the only non-trivial case is when $z \in \bigcup_{C \in \mathcal{I}_x} C^n$, $y \in \bigcap_{C \in \mathcal{I}_x} C^{n+1}$. But this implies that $z \in \tilde{C}^n$ for some cut $\tilde{C}$ including x, and so $z, y \in \tilde{C}$, which in turn implies $z \sim y$.
3. C_x^L is a cut. By contradiction, suppose $\exists z \notin C_x^L$ s.t. $z \sim y$ for all $y \in C_x^L$. Since $z \sim x$ we have that $h(z) \in \{n-1, n, n+1\}$. We distinguish these three cases:
   - if $h(z) = n-1$, then $\{x, z\}$ can be extended to a cut of height $n-1$ including x, which contradicts our first hypothesis;
   - if $h(z) = n$, then $\{x, z\}$ can be extended to a cut $C \in \mathcal{I}_x$ containing z, and so $z \in C_x^L$, a contradiction;
   - if $h(z) = n+1$, take any $C \in \mathcal{I}_x$. Now $z \sim y$ for all $y \in C^n$, and so $z \in C$. Thus $z \in C_x^L$, a contradiction.
4. It is straightforward to verify that $C_x^L \preceq C$ for all $C \in \mathcal{I}_x$.

Second case. Suppose that some cut of height $n-1$ contains x. Let:

$$C_x^L = (\bigcup_{C \in \mathcal{I}_x^{n-1}} C^{n-1}) \cup (\bigcap_{C \in \mathcal{I}_x^{n-1}} C^n).$$

and observe that:

1. $x \in C_x^L$, because $x \in C^n$ for all $C \in \mathcal{I}_x$.
2. C_x^L is a set of pairwise independent elements (straightforward).

3. C_x^L is a cut. Take a z as before:

   - if $h(z) = n-1$, then $\{x, z\}$ can be extended to a cut including z, and so $z \in C_x^L$, a contradiction;
   - if $h(z) = n$, then take any cut $C \in \mathcal{I}_x^{n-1}$. We have that $z \sim y$ for all $y \in C^{n-1}$, and so $z \in C$. Thus $z \in C_x^L$, a contradiction;
   - the case $h(z) = n + 1$ is impossible, since $\bigcup_{C \in \mathcal{I}_x^{n-1}} C^{n-1}$ is non-empty.

4. It is immediate to verify that $C_x^L \preceq C$ for all $C \in \mathcal{I}_x$. □

The following lemma gives a useful characterization of the order relation $<$ in terms of the ordering $\prec$ among cuts.

Lemma 6.4.3 *For all $x, y \in P$, we have:*

$$x < y \text{ iff } C_x^R \prec C_y^L.$$

Proof: $\Longrightarrow$ In virtue of Corollary 6.3.1, if the cut C contains z then

$$h(z) \geq h(C) \geq h(z) - 1,$$

and so

$$h(C_y^L) \geq h(y) - 1 \geq h(x) \geq h(C_x^R).$$

If $h(C_y^L) > h(C_x^R)$ the result is immediate. Suppose that $h(C_y^L) = h(C_x^R) = n$. We have three cases:

- if $C_x^R = C_y^L$ then $x \sim y$: a contradiction;
- if $C_y^L \prec C_x^R$ then $(C_y^L)^n \supset (C_x^R)^n$ and $(C_y^L)^{n+1} \subset (C_x^R)^{n+1}$. Observe that $h(x) = n$ and $h(y) = n + 1$; so $x \in (C_x^R)^n \subset C_y^L$ and thus $x \sim y$: a contradiction;
- the only possible case is thus $C_x^R \prec C_y^L$.

$\Longleftarrow$ There is no cut including both x and y, and so $x \not\sim y$. If it were $y < x$, by the first part of this proof, we would obtain $C_y^R \prec C_x^L$ and so $C_y^L \prec C_x^R$, which contradicts the hypothesis. □

Define the following relations:

$$\begin{array}{lll} x <_L y & \text{iff} & C_x^L \prec C_y^L \\ x <_R y & \text{iff} & C_x^R \prec C_y^R \end{array}$$

Observe that:

Property 6.4.1 *The relations $<_L, <_R$ are weak orders.*

Proof: We prove this for $<_L$ only (the proof for $<_R$ is analogous). It is trivial to show that $<_L$ is a partial order. Moreover, we have:

$$\begin{array}{rcl} x \sim_L y & \iff & x \not<_L y \text{ and } y \not<_L x \\ & \iff & C_x^L \not\prec C_y^L \text{ and } C_y^L \not\prec C_x^L \\ & \iff & C_x^L = C_y^L. \end{array}$$

So $\sim_L$ is transitive. □

We shall now give a different characterization of $<_L$ and $<_R$.

Lemma 6.4.4 *1. $x <_L y$ if and only if $\exists z.\, x \sim z < y$;*

2. $x <_R y$ if and only if $\exists z.\, x < z \sim y$.

Proof: (We prove only 1) $\Longleftarrow$ Using Lemma 6.4.3, from $z < y$ we obtain $C_z^R \prec C_y^L$. But $x \sim z$ implies that there exists a cut including both x and z, and so $C_x^L \preceq C_z^R$. So $C_x^L \prec C_y^L$ and thus $x <_L y$.
$\Longrightarrow$ Observe that $C_x^L \backslash C_y^L \neq \emptyset$ (if it were $C_x^L \backslash C_y^L = \emptyset$ then either $C_x^L = C_y^L$, contradicting $C_x^L \prec C_y^L$, or $C_y^L \subset C_x^L$, contradicting the fact that C_x^L is a cut). Take any $z \in C_x^L \setminus C_y^L$. Of course $z \sim x$; moreover $C_z^R \prec C_y^L$, since otherwise $z \in C_y^L$. So $z < y$ as required. □

6.5 The structure of lines

The results obtained so far allow us to prove some other interesting properties of the relation $\prec$. We will show that $(\mathcal{C}_P, \prec)$ contains a least element, and also a maximum whenever P contains a maximal element. Recall that an element $x \in P$ is called *minimal* (maximal) if and only if there is no y such that $y < x$ ($x < y$, respectively).

Observe that equivalently, an element is minimal precisely when its height is zero, and Property 6.2.2 (3) can be restated by saying that (at least) one minimal element always exists.

Property 6.5.1 *If x and x' are two minimal elements, then $C_x^L = C_{x'}^L$. Moreover, if we let $C_{\min} = C_x^L$ (for any minimal element x), we have that $C_{\min}$ is the least cut w.r.t. $\prec$.*

Proof: The first part is straightforward: in fact, suppose for example that $C_x^L \prec C_{x'}^L$. Then $x <_L x'$ and so, by Lemma 6.4.4, $x \sim z < x'$ for some z, which contradicts the minimality of x'. Now, take any cut C, and suppose by contradiction that $C \prec C_x^L$. Let y be any element of C. We would have:

$$C_y^L \preceq C \prec C_x^L \implies y <_L x$$

and so (Lemma 6.4.4) $y \sim z < x$ for some z, which contradicts the minimality of x. □

With the same technique, one could easily prove that:

Property 6.5.2 *Suppose that P contains at least one maximal element. Then, for any two maximal elements x and x', we have $C_x^R = C_{x'}^R$. Moreover, if we let $C_{\max} = C_x^R$ (for any maximal x), then $C_{\max}$ is the maximum cut w.r.t. $\prec$.* □

We shall now consider some properties of the lines in a strongly noetherian semiorder. As usual, we define the covering relation $\lessdot$ as follows:

$$x \lessdot y \text{ iff } x < y \text{ and } \nexists z.\, x < z < y.$$

Of course, the transitive closure $\lessdot^*$ of the covering relation is included in $<$, but the converse does not hold in the general case: e.g., if you consider the poset $(\mathbb{Q}, <)$, the covering relation is empty ... Nevertheless, in the case of strongly noetherian posets the two relations actually coincide.

Lemma 6.5.1 *The transitive closure of* $\lessdot$ *coincides with* $<$ *in the case of a strongly noetherian poset*[8].

Proof: Suppose that $x < y$. If $x \lessdot y$ we are done; otherwise, take any line L including both x and y, and consider:

$$A = \{z : z \in L, x \leq z \leq y\}.$$

Since A is totally ordered and has a maximum, A is finite (because P is strongly noetherian), say $A = \{x = z_0 < z_1 < \ldots < z_n = y\}$. Moreover $z_i < z_{i+1}$ are really covering relations, as one can easily verify. So we have that $x \lessdot^* y$ as required. □

Now, one easily sees that a line L in a strongly noetherian poset can be one of two kinds:

1. a sequence of the form $L = \{x_0 \lessdot x_1 \lessdot \ldots \lessdot x_n\}$, where x_0 is minimal and x_n is maximal;

2. an infinite sequence of the form $L = \{x_0 \lessdot x_1 \lessdot \ldots\}$ where x_0 is minimal.

Also notice that all the sequences in a certain semiorder will be of the same kind.

6.6 Connectedness and K-density

Before going on with our formal treatment of semiorders, we want to stop and return for a moment to the example of Section 5.1. Most of us would prefer a cup with one spoon of sugar (say 5 grams) to a cup with five spoons; yet, if we use c_α to denote a cup with α grams of sugar, we would certainly be indifferent between c_α and $c_{\alpha+1/100}$. So:

$$c_5 \sim c_{5.001} \sim c_{5.002} \sim \cdots \sim c_{24.999} \sim c_{25}$$

but still c_5 is *not* indifferent from c_{25}. We can pass from c_5 to c_{25} through an "indifference" chain, i.e., a chain of pairwise indifferent objects. This resembles the classical notion of continuum developed in mathematics; as Brouwer says (quoted by Weyl in [Wey49]), speaking about continua,

> The separateness of two places, upon moving them toward each other, slowly and in vague gradation passes over into indiscernibility.

[8]This property is usually called "combinatoriality".

The fact that we can move from one object to another object distinguishable from the former through a chain of indistinguishable elements is one of the typical paradoxes of continuum (as Russell [Rus37] says, *the arrow, at every moment of flight, is truly at rest*). What is strange and peculiar, here, is the fact that our indifference chain is finite, which hurts the common sense, as Borel [Bor46] states:

> Une ligne droite apparaît ainsi comme une suite continue de points, sans aucun vide, que l'on ne peut parcourir qu'en passant successivement par tous les points, dont le nombre est nécessairement infini, car un nombre fini de points, si grand qu'il soit, laisserait nécessairement des intervalles vides séparant ces points.

Yet, Petri [Pet96] observes that Poincaré, Weyl, Dirac et al. have pointed out how, ideally, the mathematics of physical theory should be "combinatorial", which should not be misunderstood as implying "discrete": as a matter of fact, one can think of discrete, or even finite, mathematical structures which exhibit a continuous behaviour, and Petri asserts that this assumption is *fully compatible with the mathematical view that the continuum of real numbers can and should be chosen as [the paradigm for continua].* The problem of filling the gap between discrete and continuous models has been extensively studied, bearing to a hierarchy of possible solutions (see [Pet80b]).

Coming back to our measurement problem, our current purpose is then to design a system by which measurement can be achieved in a continuous fashion. In other words, we want to set up a family of reference measurands in such a way that the resolution power of our measuring tools is exploited. A way to do this is to assume that any two measurands are joined by an indifference chain, witnessing the fact that no gap exists between their measures; in fact, as Smith [Smi93] shows, this is a sufficient condition for obtaining that any given measurand (within a certain range) is matched by one of the references.

This condition on semiorders is called connectedness[9]; a poset $(P, <)$ is *connected* if and only if $\sim^* = P^2$ (where $\sim^*$ denotes as usual the transitive closure of $\sim$). In other words, there exists an indifference chain between any two elements of P, or equivalently the associated tolerance space is a connected graph.

Connectedness in a strongly noetherian semiorder turns out to be equivalent to a property of "continuity" in the dynamics of cuts. In practice, a semiorder is connected if and only if we never have to "jump over a gap" when moving from a cut to the subsequent one, in a way which is made precise by the following theorem.

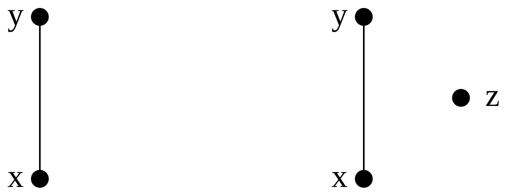
Theorem 6.6.1 *Let $(P, <)$ be a strongly noetherian semiorder. The following are equivalent:*

(i) P is connected;

(ii) for all $C_1, C_2 \in \mathcal{C}_P$

$$C_1 \prec\!\!\cdot\, C_2 \text{ implies } C_1 \cap C_2 \neq \emptyset;$$

[9] In [Bol96] we used the word "coherence" instead; although this is the term used in measurement theory and social sciences, we have preferred to adopt another name here, not to clash with the use of the word "coherent" in domain theory, see Chapter 3.



(a) (b)

Figure 6.8: (a) A disconnected and (b) a connected semiorder

(iii) for all $n < n_P = \sup\{h(x), x \in P\}$*, there exists a vertical cut of height* n*.*

Proof: (i) $\implies$ *(ii)* Suppose that $C_1 \prec\!\!\cdot\, C_2$ and $C_1 \cap C_2 = \emptyset$. Then we have that:

Claim:

1. if $x \in C_1$ and $x \sim^* z$ then $C_z^R \preceq C_1$;
2. if $y \in C_2$ and $y \sim^* z$ then $C_2 \preceq C_z^L$.

Proof of Claim: (Only part 1) By induction on the length of the indifference chain; if $x = z$ then $C_z^R = C_1$.

If $x \sim^* w \sim z$ then, by induction hypothesis $C_w^R \preceq C_1$. Since $w \sim z$ we have $C_z^L \preceq C_w^R$. If it were $C_1 \prec C_z^R$ then $C_2 \preceq C_z^R$ and so, since $C_z^L \preceq C_w^R \preceq C_1 \prec C_2 \preceq C_z^R$ we would have $z \in C_1 \cap C_2$. Thus $C_z^R \preceq C_1$ *Q.E.D. Claim*

Now let $x \in C_1, y \in C_2$. If $x \sim^* y$ then, by the Claim, $C_y^R \preceq C_1$ and $C_2 \preceq C_x^L$ and so $C_y^R \prec C_x^L$. But $C_x^L = C_1, C_y^R = C_2$ and so $C_2 \prec C_1$: a contradiction.

(ii) $\implies$ *(iii)* Suppose by contradiction that no vertical cut of height $m < n_P$ exists. Let $C_2 = P^{m+1}$; of course C_2 is non empty (since $m + 1 \leq n_P$) and moreover C_2 is a cut: if there was a $z \notin C_2$ such that z is incomparable with any element of C_2, then $h(z) = m$ and we would find a vertical cut of height m. Now let C_1 be any cut extending P^m (only one such cut exists, actually). Since C_2 is the first cut of height $m + 1$, we have $C_1 \prec\!\!\cdot\, C_2$; but $C_1 \cap C_2 = \emptyset$, which contradicts (ii).

(iii) $\implies$ *(i)* If (iii) holds, for all $n < n_P$ there exist two elements $x_n, y_n \in P$ s.t. $h(x_n) = n, h(y_n) = n + 1$ and $x_n \sim y_n$. Let $x, y \in P$ with $h(x) = p \leq h(y) = q$, and consider the sequence $x, x_p, y_p, \dots, x_{q-1}, y_{q-1}, y$. Using the hypothesis, we have that this is a (finite) indifference chain. $\square$

We have introduced connectedness as a desirable property of our measuring system, and shown how this is equivalent to a property of continuity in the structure of cuts. Fig. 6.8 (a) shows a non-connected semiorder with a "gap" between two subsequent cuts (in fact $\{x\} \cap \{y\} = \emptyset$); in Fig. 6.8 (b) we have no gaps (in fact $\{x, z\} \prec\!\!\cdot\, \{z, y\}$ and $\{x, z\} \cap \{z, y\} = \{z\}$), and the semiorder is therefore connected, as one easily verifies.

Even though everyone agrees about the necessity of a notion like connectedness for a semiorder to be "continuous", connectedness is hardly enough for having all the desirable properties of a continuum stored in our discrete setting. Another requirement is that known as K-density [Smi93, Pet96], a term coming after Petri's work on the study of concurrent processes.

In order to motivate K-density as a basic property of continuous structures, we want to recall briefly Carnap's work on the axiomatization of space-time topology [Car58].

Carnap develops an axiom system for treating the motions of (idealized, unextended) physical particles; he takes as individuals the moments of particles (called *world-points*) each assigned to a space-time point. The set of all world-points of a particle is called a *world-line*, and it is totally ordered by a relation which is interpreted as "being earlier than", a *local time order.* For taking into account the interaction between particles, a *coincidence* relation is defined among world-points: two world-points coincide when they are assigned the same space-time position.

This, together with the local ordering on world-lines, induces a partial order S on the set of world-points, which can be interpreted as the possibility for an effect to reach a world-point from another. Now, in this partial order, the incomparability relation I determines the simultaneity of world-points, in the sense of Reichenbach, which defines two world-point simultaneous if and only if no signal can ever go from one to the other.

Now, a maximal antichain (cut, in our terminology) of S is what Carnap calls a space, "a three-dimensional cross section of the four-dimensional space-time world, the sectioning being done across the time direction" [Car58]. Similarly, a maximal chain (line) of S is a signal line in the sense of Carnap, and determines the evolution of a process in the space-time world. What one should require for time to be continuous is that each single particle is somewhere in each moment, which is formalized as the assumption that each signal line intersects each space in (exactly) one point[10].

This property is known as K-density; more formally, a poset $(P, <)$ is *K-dense* iff

$$\forall L \in \mathcal{L}_P \forall C \in \mathcal{C}_P.\, L \cap C \neq \emptyset.$$

As far as measurement is concerned, K-density is a property related to the granularity of the set of reference measurands [Smi93]. K-dense posets (also known as chain-antichain-complete posets) are also a well-studied subject of the combinatorial theory of orders [Gri69, LM73, EZZ86, Riv86].

An interesting characterization can be given for K-density in the case of strongly noetherian semiorders: it turns out that K-dense semiorders are precisely those where the covering relation is reflected in the ordering of cuts.

Theorem 6.6.2 *Let $(P, <)$ be a strongly noetherian semiorder. The following are equivalent.*

(i) P is K-dense;

(ii) for all $x, y \in P$, $x \lessdot y$ implies $C^R_x \prec\!\!\cdot\, C^L_y$.

Proof: (i) $\implies$ (ii) Suppose that $x \lessdot y$ and $C^R_x \prec C \prec C^L_y$. Let now L be any line with $x, y \in L$; the set of cuts which are intersected by L is:

$$T = \cup_{w \in L}[C^L_w, C^R_w]$$

[10] This is axiom 48.D6 of [Car58].

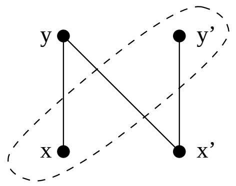


Figure 6.9: An N-shaped poset (the sloping line is a covering relation)

where $[C_1, C_2] = \{C : C_1 \preceq C \preceq C_2\}$. P is K-dense, and so $T = \mathcal{C}_P$; in particular $C \in T$, i.e. there exists a $z \in L$ such that $C_z^L \preceq C \preceq C_z^R$. If we had $C_x^R \prec C_z^L$ and $C_z^R \prec C_y^L$, then we would have $x < z < y$ contradicting $x \lessdot y$. So it must be:

$$C_z^L \preceq C_x^R \text{ or } C_y^L \preceq C_z^R.$$

But $C_z^L \preceq C_x^R \prec C \preceq C_z^R$ implies $x \sim z$, and the other case leads similarly to $y \sim z$: both cases are impossible since L is a line.
(ii) $\Longrightarrow$ *(i)* Straightforward, for the characterization of the lines in a strongly noetherian poset. □

The above result allows us to obtain a local condition which is equivalent to K-density for strongly noetherian semiorders. If $(P, <)$ is a poset, an N of P is a quadruple $x, y, x', y' \in P$ such that $x \lessdot y, x < x', y' < y$ and no other pair is comparable; the simplest poset containing an N is that represented in Fig. 6.9, which explains the name "N".

The cut formed by the circled elements does not intersect the line $\{x', y\}$. The following property, which is a corollary to the previous theorem, states that N-freeness is equivalent to K-density in the case of strongly noetherian semiorders.

Property 6.6.1 *A strongly noetherian semiorder is K-dense iff it is N-free.*

Proof: Suppose that $(P, <)$ is *not* K-dense: we will find an N in P. Using Theorem 6.6.2, we find $x, y \in P$ such that $x \lessdot y$ and $C_x^R \prec C \prec C_y^L$ for some C. Now $x, y \notin C$ and so there exist $x', y' \in C$ such that $x \not\sim x', y \not\sim y'$, and of course we shall have $x < x', y' < y$. Also $x' \sim y'$ (since x', y' belong to the same cut). Moreover $x' \sim y$ (if $x' \not\sim y$, then $x' < y$ which in turn implies $x < x' < y$, contradicting $x \lessdot y$) and similarly $x \sim y'$. So x, y, x', y' is an N of P. □

Chapter 7

Conclusions and further work

The original motivation for our work was a reconstruction from below of Petri's axiomatization of concurrency. Clearly, the results presented here are just the first step towards this goal. Here are some research directions which are left open by this thesis; some of these topics will be hopefully covered, at least in part, in the final version.

- *Universal constructions.* We think that the general techniques used here to give an explicit deterministic construction for many sorts of representations (Section 4.3) can be further generalized, e.g., in order to obtain a universal homogeneous general event structure (although, in that case, the associated domain could not be homogeneous, because of the negative results of [DG93]). The same can be done also for the probabilistic constructions of Section 4.6. An interesting open question related to the universal constructions is whether the universal dI-domain (coherent dI-domain) obtained in Corollary 4.3.4 (Theorem 4.5.5, respectively) is homogeneous; if the answer to this question is negative, we could wonder whether it is possible to use the same number-theoretic techniques for obtaining *directly* a homogeneous domain, rather than working on representations.
  An important, and probably difficult, question is whether it is possible to have suitable universal constructions also for some full subcategories of **TolSp**, e.g., for the subcategory of all tolerance spaces induced by partial orders (interval orders, semiorders ...). This problem is likely to require some additional special techniques, because those categories are not algebroidal anymore, and so saturation cannot be applied (at least, not in a direct way).

- *Approximation techniques.* The idea of approximating topological spaces by using two (inverse) limits (one in the category of tolerance spaces, and the other in the category of topological spaces) is still in the early stages. We believe it may be given a more systematic setting, but at present it is nothing more than a temptative technique, even though the results obtained so far are promising. We also believe that relating such approaches with other similar ones proposed in the literature (see, e.g., [Smy92],[Sün95]) could be very fruitful in order to refine our constructions. Also, a comparison with [Smy95] would be interesting, although the results in that paper are obtained in a different setting, and this makes it difficult to appreciate the similarities with ours.

- *Stone duality and approximation sequences.* An interesting open problem is to characterize those topological spaces which are (homeomorphic to) the reduced limit of a discretely topologized approximation sequence of finite tolerance spaces (we have presented some examples of such spaces in Sections 5.7, 5.8 and 5.9). A clue for solving this problem is offered by the well-known Stone duality, which could be expressed (as Smyth suggests [Smy92][Proposition 5.0.18]) as follows: a second-countable space is a Stone space if and only if it is the inverse limit (in the category **Top**) of a sequence of discrete finite spaces. Thus, approximable spaces are just "special" quotients of second-countable Stone spaces.
  A related open problem is to characterize the topological spaces which can be obtained as a limit of an effectively presentable approximation sequence. This question is clearly central if we aim at applying our results in a computational setting.

- *Generalizations of semiorders.* The study of tolerance spaces induced by semiorders could be probably generalized in various ways, for example by considering different kinds of orders (ranging from the n-semiorders [Bog82], to interval orders [Fis85], up to general partial orders); probably, this would result in very interesting characterizations of chains and antichains (cliques and independent sets) for large classes of tolerance spaces induced by partial orders.

- *Applications to concurrency theory.* Another serious drawback with semiorders is that many posets occurring in physics and in control theory, which can be used to denote measuring scales, are not semiorders. Two examples are represented by the so-called *time orthoids* (defined by Petri [Pet96]), which can be used as a concrete image of the ticking of a clock, and by the combinatorial continuum with inner points only, defined in [PS87]. Moreover, if one wants to apply combinatorial results to control and concurrency theory, the notion of poset is too restrictive because it does not allow cyclicity to be taken into account. In this case, cyclic structures like those defined in [Pet80a] should be considered: a cyclic structure is defined by means of a quaternary separation relation, similar to that used in projective geometry. A generalization in this direction of the results obtained in Chapter 6 is desirable, but it requires a more thorough understanding of the relations existing among cyclic orders and posets.

Finally, on a more philosophical side, we would be interested in pursuing the theme of this thesis up to gaining a more thorough understanding of Petri's ideas on finite models for continuous phenomena, perhaps also in the light of the contributions of A.W. Holt [Hol80, Hol74b, Hol74a].

Appendix A

Background

A.1 Category theory

We introduce the fundamental notions of category theory used in this thesis; most of them are quite standard, and can be found in any textbook of categories, like [Mac71].
A *category* $\mathcal{C}$ is defined by the following data:

- a class $\mathrm{Obj}(\mathcal{C})$ of *objects*;
- a class $\mathrm{Arr}(\mathcal{C})$ of *arrows* (or morphisms);
- two maps $\mathrm{dom}, \mathrm{cod} : \mathrm{Arr}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C})$, giving, for each arrow f of $\mathcal{C}$, two objects, called the *domain* and *codomain* of f; we write $f : A \to_{\mathcal{C}} B$ to mean that f is an arrow of $\mathcal{C}$ with domain A and codomain B (the subscript is omitted whenever it can be deduced from the context);
- a partial composition $\circ : \mathrm{Arr}(\mathcal{C}) \times \mathrm{Arr}(\mathcal{C}) \to \mathrm{Arr}(\mathcal{C})$, satisfying the following constraints: $f \circ g$ is defined if and only if $\mathrm{cod}(g) = \mathrm{dom}(f)$; moreover, $(f \circ g) \circ h = f \circ (g \circ h)$ whenever both sides are defined;
- for each $A \in \mathrm{Obj}(\mathcal{C})$, there is an arrow $\mathbf{1}_A : A \to_{\mathcal{C}} A$ such that, for any arrow f (g) of $\mathcal{C}$ with domain (codomain) A, it holds that $f \circ \mathbf{1}_A = f$ $(\mathbf{1}_A \circ g = g)$.

There are many examples of categories; here we list some of them:

- **Set**: the category of sets, with functions as arrows, and with composition interpreted as the standard function composition;
- **Mon**: the category of monoids, with monoid morphisms as arrows, and with the usual function composition;
- **Top**: the category of topological spaces, with continuous functions as arrows.

All these categories are *large*, in the sense that their object class is a proper class and not a set; in many other cases, both objects and arrows form a set, and then we speak of *small* categories. As an example, every preordered set P, with preorder $\sqsubseteq$, can be seen as a small category whose objects are simply the elements of P, and with an arrow from

A to B if and only if $A \sqsubseteq B$. An object $A \in \mathrm{Obj}(\mathcal{C})$ is *weakly initial* iff for any object B there is an arrow from B to A. Clearly, in the case of a category representing a poset, a weakly initial[1] object is simply the minimum.

For each two objects $A, B \in \mathrm{Obj}(\mathcal{C})$, we let $\mathrm{Hom}_{\mathcal{C}}(A, B)$ be the class of arrows from A to B. A *subcategory* of $\mathcal{C}$ is a category $\mathcal{D}$ whose objects (arrows) are a subclass of those of $\mathcal{C}$, with domain, codomain, compositions and identities defined by restricting the corresponding data of $\mathcal{C}$. We say that the subcategory is *full* if, for every $A, B \in \mathrm{Obj}(\mathcal{D})$ we have $\mathrm{Hom}_{\mathcal{D}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$ (i.e., if we look at a category as simply a directed graph, $\mathcal{D}$ is an induced subgraph of $\mathcal{C}$). An arrow $f : A \to_{\mathcal{C}} B$ is

- an *isomorphism* if there exists an arrow $g : B \to_{\mathcal{C}} A$ such that $g \circ f = \mathbf{1}_A$ and $f \circ g = \mathbf{1}_B$; in this case, we say that A and B are isomorphic, and write $A \cong B$;
- *monic* if, for any two arrows $g, h : C \to_{\mathcal{C}} A$, if $f \circ g = f \circ h$ then $g = h$;
- *epi* if, for any two arrows $g, h : B \to_{\mathcal{C}} C$, if $g \circ f = h \circ f$ then $g = h$.

The *skeleton* of $\mathcal{C}$ is the set of equivalence classes of objects of $\mathcal{C}$ w.r.t. isomorphism, i.e., $\mathrm{Obj}(\mathcal{C})/\cong$.

Let $\mathcal{C}, \mathcal{D}$ be two categories; a *functor* $F : \mathcal{C} \to \mathcal{D}$ maps objects of $\mathcal{C}$ to objects of $\mathcal{D}$, and arrows of $\mathcal{C}$ to arrows of $\mathcal{D}$, in such a way that:

- for all $f \in \mathrm{Arr}(\mathcal{C})$ we have $\mathrm{dom}(F(f)) = F(\mathrm{dom}(f))$ and $\mathrm{cod}(F(f)) = F(\mathrm{cod}(f))$;
- for all $A \in \mathrm{Obj}(\mathcal{C})$ we have $F(\mathbf{1}_A) = \mathbf{1}_{F(A)}$;
- if $f : A \to_{\mathcal{C}} B$ and $g : B \to_{\mathcal{C}} C$ then $F(g \circ f) = F(g) \circ F(f)$.

In particular, for any category $\mathcal{C}$, one can define the *identity functor* $\mathbf{1}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ which acts as the identity both on objects and on arrows. Clearly, we can compose functors by composing their object- and arrow-component separately; thus, we obtain a (large) category, whose objects are all small categories, and whose arrows are functors, with the just defined functor composition (this category is usually denoted by **Cat**).

A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *full* iff for every pair $c, c' \in \mathrm{Obj}(\mathcal{C})$ and every arrow $g : F(c) \to_{\mathcal{D}} F(c')$, there is an arrow $f : c \to_{\mathcal{C}} c'$ such that $g = F(f)$. It is *faithful* iff for every pair $c, c' \in \mathrm{Obj}(\mathcal{C})$ and every two arrows $f, g : c \to_{\mathcal{C}} c'$, if $F(f) = F(g)$ then $f = g$.

Let now $F, G : \mathcal{C} \to \mathcal{D}$ be two functors; a *natural transformation* $\tau : F \dot{\to} G$ is a function which assigns, to each object c of $\mathcal{C}$, an arrow $\tau_c : F(c) \to_{\mathcal{D}} G(c)$ such that, for every arrow $f : c \to_{\mathcal{C}} c'$, the following diagram is commutative[2]:

$$\begin{array}{ccc} F(c) & \xrightarrow{\tau_c} & G(c) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle G(f)} \\ F(c') & \xrightarrow{\tau_{c'}} & G(c') \end{array}$$

[1]In this case, really an initial object.

[2]This means that, for any two paths in the diagram having the same starting node and the same ending node, the strings of compositions obtained on the two paths are equal.

The arrows of $\mathcal{D}$ which are in the image of τ are called the *components* of the transformation; we say that the transformation is a *natural isomorphism*, and write $\tau : F \stackrel{\cdot}{\cong} G$, or simply $F \stackrel{\cdot}{\cong} G$, if every component is an isomorphism.

Given two categories $\mathcal{C}$ and $\mathcal{D}$, we say that they are *equivalent* if there exist two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \stackrel{\cdot}{\cong} \mathbf{1}_{\mathcal{C}}$ and $F \circ G \stackrel{\cdot}{\cong} \mathbf{1}_{\mathcal{D}}$. It is folklore that

Remark A.1.1 *Let Φ be a statement which can be expressed using only categorical notions; if Φ holds for the category $\mathcal{C}$, and if $\mathcal{D}$ is equivalent to $\mathcal{C}$, then the statement also holds for $\mathcal{D}$. Moreover, if the statement implies the existence of an object (arrow) A (f, resp.) of $\mathcal{C}$ satisfying certain properties, and if the equivalence between $\mathcal{C}$ and $\mathcal{D}$ is given by the two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, then $F(A)$ ($F(f)$, resp.) is an object (arrow, resp.) of $\mathcal{D}$ satisfying the same properties.*

This remark was used implicitly in the proof of existence of the universal homogeneous (atomic, coherent) dI-domain, in Chapter 4. The following well-known result is often used to prove categorical equivalence:

Theorem A.1.1 (MacLane [Mac71], Theorem IV.4.1) *The following properties of a functor $F : \mathcal{C} \to \mathcal{D}$ are equivalent:*

1. *F gives an equivalence of categories;*
2. *F is full and faithful, and each object $d \in Obj(\mathcal{D})$ is isomorphic to $F(c)$ for some object c of $\mathcal{C}$.* □

An *ω-chain* [DG93], in a category $\mathcal{C}$, is a sequence[3] $(A_i, f_i)_{i\in\omega}$ where each A_i is an object of $\mathcal{C}$, and $f_i : A_i \to_{\mathcal{C}} A_{i+1}$ is an arrow of $\mathcal{C}$. The *colimit* (or direct limit) of such a chain is given by an object A and a sequence of arrows $\psi_i : A_i \to_{\mathcal{C}} A$, such that

1. for every $i \in \omega$, we have $f_i \circ \psi_i = \psi_{i+1}$ (a sequence satisfying this requirement is usually called a *cone*);
2. if B is an object of $\mathcal{C}$, with a cone $\phi_i : A_i \to_{\mathcal{C}}$, there is a unique arrow $h : A \to B$ such that $h \circ \phi_i = \psi_i$.

The first part of this definition can be sketched by the following commutative diagram:

$$\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[d, "\psi_0"'] & A_1 \arrow[r, "f_1"] \arrow[dl, "\psi_1"] & A_2 \arrow[r, "f_2"] \arrow[dll, "\psi_2"] & A_3 \cdots \\
A
\end{tikzcd}$$

It is clear that the colimit is unique up to isomorphisms, if it exists, and we write $A = \varinjlim(A_i, f_i)$. If $\mathcal{C}$ represents (the categorical version of) a partially ordered set, an ω-chain is interpreted as a standard ascending countable chain, and its colimit (if it exists) is the least upper-bound.

[3]Or, equivalently, a functor $F : \omega \to \mathcal{C}$, where ω is considered as (the categorical representation of) the poset of natural numbers.

Conversely, an *ω-cochain* in a category $\mathcal{C}$, is a sequence[4] $(A_i, f_i)_{i\in\omega}$ where each A_i is an object of $\mathcal{C}$, and $f_i : A_{i+1} \to_{\mathcal{C}} A_i$ is an arrow of $\mathcal{C}$. The *limit* (or inverse limit) of such a chain is given by an object A and a sequence of arrows (called the *cone*) $\psi_i : A \to_{\mathcal{C}} A_i$, such that

1. for every $i \in \omega$, we have $f_i \circ \psi_{i+1} = \psi_i$;

2. if B is an object of $\mathcal{C}$, with a cone $\phi_i : B \to_{\mathcal{C}} A_i$ satisfying the previous statement, then there is a unique arrow $h : B \to A$ such that $\psi_i \circ h = \phi_i$.

Also, the limit is unique up to isomorphisms, if it exists, and we write $A = \underleftarrow{\mathrm{Lim}}(A_i, f_i)$. In the case of posets, an ω-cochain is a countable descending chain, and its limit is the greatest lower bound.

An object B in a category $\mathcal{C}$ is *finite* (according to [DG93]) if, whenever $(A_i, f_i)_{i\in\omega}$ is an ω-chain with $\underrightarrow{\mathrm{Lim}}(A_i, f_i) = A$ and with cone $\psi_i : A_i \to A$, and $g : B \to A$ is an arrow, there exists an index n such that there is a unique arrow $h : B \to A_n$ satisfying that $g = \psi_n \circ h$. Diagrammatically:

$$\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[d, "\psi_0"'] & A_1 \arrow[r, "f_1"] \arrow[dl, "\psi_1"] & A_2 \arrow[r, "f_2"] \arrow[dll, "\psi_2"] & A_3 & \cdots & A_n \arrow[dlllll, "\psi_n"] & \cdots \\
A & & & & & & \\
B \arrow[u, "g"'] \arrow[uurrrrr, "\exists! h"'] & & & & & &
\end{tikzcd}$$

The full subcategory of finite objects is denoted by $\mathcal{C}_f$.

A.2 Measure theory

We introduce here the very basic notions of (abstract) measure theory we used in this thesis; the notation and definitions are standard, and we refer to [Doo94] for more on this topic.

A family $\mathbb{A}$ of subsets of a set X is called a *σ-algebra* on X if it satisfies the following constraints:

1. $X \in \mathbb{A}$;

2. if $M \in \mathbb{A}$ then also $M^C \in \mathbb{A}$ (the complement being taken with respect to the whole set X);

3. if $\mathbb{S} \subseteq \mathbb{A}$ is a countable set, then also $\cup\mathbb{S} \in \mathbb{A}$.

In other words, a σ-algebra contains the whole set, and it is closed with respect to complements and countable unions. The pair $(X, \mathbb{A})$ is often called *measure space*, and the elements of $\mathbb{A}$ are called *measurable sets.*

Here are some properties of a measure space $(X, \mathbb{A})$:

[4] Or, equivalently, a functor $F : \omega^{op} \to \mathcal{C}$, where ω^{op} is like ω but with all arrows reversed.

- the empty set is measurable, i.e., $\emptyset \in \mathbb{A}$;
- $\mathbb{A}$ is closed under (finite and) countable unions and intersections; i.e., if $\mathbb{S} \subseteq \mathbb{A}$ is (finite or) countable, then $\cup\mathbb{S} \in \mathbb{A}$ and $\cap\mathbb{S} \in \mathbb{A}$;
- $\mathbb{A}$ is closed under set-difference (i.e., if $M_1, M_2 \in \mathbb{A}$ then also $M_1 \setminus M_2 \in \mathbb{A}$).

Two measurable sets $M_1, M_2 \in \mathbb{A}$ are *orthogonal* if and only if $M_1 \cap M_2 = \emptyset$. The following property states that every countable limit of measurable sets can always be expressed as a limit of pairwise orthogonal measurable sets.

Property A.2.1 *Let $\langle A_n \rangle_{n\in\omega}$ be a sequence of measurable sets in the measure space $(X, \mathbb{A})$. Then, there is a sequence $\langle B_n \rangle_{n\in\omega}$ of measurable sets such that*

- $\cup_{n\in\omega} A_n = \cup_{n\in\omega} B_n$;
- *for all $n \in \omega$ we have $B_n \subseteq A_n$;*
- *for all $n, m \in \omega$ $(n \neq m)$ the sets B_n and B_m are orthogonal.*

Proof: (Sketch) Just take $B_n = A_n \setminus (A_0 \cup A_1 \cup \ldots \cup A_{n-1})$. □

Here is an easy, but important, property of σ-algebras:

Lemma A.2.1 *Let $\mathcal{A}$ be a non-empty set of σ-algebras on X. Then $\cap\mathcal{A}$ is a σ-algebra on X.* □

As a consequence, we have:

Corollary A.2.1 *Let $\mathcal{F} \subseteq \wp(X)$ be a set of subsets of X; then, there exists a σ-algebra $\mathbb{A}_\mathcal{F}$ on X such that $\mathcal{F} \subseteq \mathbb{A}_\mathcal{F}$ and moreover, if $\mathbb{A}$ is a σ-algebra on X such that $\mathcal{F} \subseteq \mathbb{A}$, then also $\mathbb{A}_\mathcal{F} \subseteq \mathbb{A}$.*

Proof: Let $\mathcal{A} = \{\mathbb{A} : \mathbb{A}$ is a σ-algebra on X, and $\mathcal{F} \subseteq \mathbb{A}\}$ and define $\mathbb{A}_\mathcal{F} = \cap\mathcal{A}$. This is still a σ-algebra, by Lemma A.2.1. □

The σ-algebra $\mathbb{A}_\mathcal{F}$ is called the "σ-algebra generated by $\mathcal{F}$", and it is in practice the least σ-algebra containing all the elements of $\mathcal{F}$ as measurable sets.

A (positive) *measure* on the measure space $(X, \mathbb{A})$ is a function $\mu : \mathbb{A} \to \mathbb{R} \cup \{\infty\}$ such that

1. for all $M \in \mathbb{A}$, $\mu(M) \geq 0$;
2. if $\langle A_n \rangle_{n\in\omega}$ is a sequence of pairwise orthogonal measurable sets, then
$$\mu(\cup_{n\in\omega} A_n) = \sum_{n\in\omega} \mu(A_n);$$
3. there exists an $M \in \mathbb{A}$ such that $\mu(M) < \infty$.

We say that μ is *normalized* (or: "a probability measure") iff $\mu(X) = 1$. The following proposition gives the main property of measure functions:

Proposition A.2.1 *Let $(X, \mathbb{A})$ be a measure space, with measure μ. Then*

1. $\mu(\emptyset) = 0$;

2. *if $M_1, M_2 \in \mathbb{A}$ and $M_1 \subseteq M_2$, then $\mu(M_1) \leq \mu(M_2)$;*

3. *if $\langle A_n \rangle_{n\in\omega}$ is a sequence of measurable sets, then*

$$\mu(\cup_{n\in\omega} A_n) \leq \sum_{n\in\omega} \mu(A_n);$$

4. *if $\langle A_n \rangle_{n\in\omega}$ is a sequence of measurable sets with $A_n \subseteq A_{n+1}$, then*

$$\mu(\cup_{n\in\omega} A_n) = \lim_{n\to\infty} \mu(A_n);$$

5. *if $\langle A_n \rangle_{n\in\omega}$ is a sequence of measurable sets with $A_{n+1} \subseteq A_n$, then*

$$\mu(\cap_{n\in\omega} A_n) = \lim_{n\to\infty} \mu(A_n);$$

6. *if μ is normalized, and $M \in \mathbb{A}$, then $\mu(M^C) = 1 - \mu(M)$;*

7. *if $\langle A_n \rangle_{n\in\omega}$ is a sequence of measurable sets with $\mu(A_n) = 0$ for all $n \in \omega$, then also $\mu(\cup_{n\in\omega} A_n) = 0$; i.e., a countable union of sets of measure zero still has measure zero.*

Proof: See [Doo94]. □

One last point which needs to be explained is independence. If μ is a (normalized) measure on $(X, \mathbb{A})$, and $A, B \in \mathbb{A}$, we say that A, B are (μ)-independent iff $\mu(A \cap B) = \mu(A)\mu(B)$.

A.3 General topology

We introduce here the fundamentals of general (point-set) topology used in this thesis; some more specific topics are introduced directly in the text. For more information on this subject, we refer the reader to [Gaa64].

A *topology* on a set X is a family $\Omega \subseteq \wp(X)$ of subsets of X such that

1. $\emptyset \in \Omega$ and $X \in \Omega$;

2. if $O_1, O_2 \in \Omega$, then also $O_1 \cap O_2 \in \Omega$;

3. if $\langle O_i \in \Omega \rangle_{i\in I}$, then also $\cup_{i\in I} O_i \in \Omega$.

In other words, a topology is a family of sets which contains the empty set, the whole set (universe), and is closed under finite intersections and arbitrary unions. The elements of Ω are called *open sets*, and the pair (X, Ω) is often called a *topological space*. Often, when speaking of a topological space (X, Ω), we simply use the letter X and leave the topology Ω unspecified (when no confusion may arise).

Note that one can order the topologies on a set X by set-inclusion; if $\Omega_1 \subseteq \Omega_2$ are two topologies on X, we often say that Ω_1 is *weaker* (or coarser) than Ω_2, and that Ω_2 is *stronger* (or finer) than Ω_1. Clearly, there is one weakest topology on X, which is $\{\emptyset, X\}$ (called the *indiscrete*, or non-discrete, one), and one strongest topology, which is $\wp(X)$ (called the *discrete* one).

A set $C \subseteq X$ is *closed* iff its complement C^C is open. Note that $\emptyset, X$ are closed sets, and that finite unions and arbitrary intersections of closed sets are also closed. A set which is both open and closed is sometimes called *clopen* ($\emptyset$ and X are the trivial clopen sets).

Here are some very basic definitions of general topology; let (X, Ω) be a topological space, and $A \subseteq X$:

- the *interior* of A is the biggest open set contained in A, i.e.,

$$\mathrm{Int}(A) = \cup\{O \in \Omega : O \subseteq A\};$$

- the *closure* of A is the smallest closed set containing A, i.e.,

$$\overline{A} = \cap\{C : A \subseteq C, C^C \in \Omega\};$$

- the *border* of A is the set of points which are contained in the closure of A but not in its interior, i.e.,

$$\mathrm{Bd}(A) = \overline{A} \setminus \mathrm{Int}(A);$$

- the *exterior* of A is the interior of its complement, i.e., $\mathrm{Ext}(A) = \mathrm{Int}(A^C)$;

- a point $x \in X$ is an *accumulation* point of A iff every open set containing x contains at least one point of A different from x; the set of all accumulation points of A (often called the "derived set") is denoted by ∂A.

One usual way of determining a topology on a set X is the following. Let $\mathcal{B} \subseteq \wp(X)$ be a family of subsets of X such that:

1. for all $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$;

2. if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

The family $\mathcal{B}$ is called a *base* (if it satisfies only axiom (1), we call it a *subbase*).

Then, let Ω be the set of subsets of X which are unions of elements of $\mathcal{B}$: this is actually a topology, called the "topology generated by the base $\mathcal{B}$". It is indeed the weakest topology including $\mathcal{B}$. In the case of a subbase, we must consider first the base obtained by taking the finite intersections of the elements of the subbase, and then generate the topology from this base.

Using (sub)bases for defining topologies is a very useful way of describing topological spaces. One important example is the following: for any two real numbers $a, b \in \mathbb{R}$, with $a < b$, let (a, b) denote the open interval with extreme a and b (i.e., the set $\{r \in \mathbb{R} : a < r < b\}$). The family of all such open intervals forms a base, and the generated topology on $\mathbb{R}$ is called the *Euclidean (or standard) topology* of $\mathbb{R}$.

Here is a necessary and sufficient condition for two bases to generate the same topology:

Theorem A.3.1 (Gaal [Gaa64], Theorem I.4.1) *Let X be a set, and $\mathcal{B}_1, \mathcal{B}_2$ be two bases on X. Let also Ω_i be the topology generated by $\mathcal{B}_i$ ($i = 1, 2$). Then Ω_1 is stronger than Ω_2 if and only if for all $B_2 \in \mathcal{B}_2$ and all $x \in B_2$, there is a $B_1 \in \mathcal{B}_1$ such that $x \in B_1 \subseteq B_2$.* □

In particular, observe that Ω_1 is stronger than Ω_2 whenever $\mathcal{B}_2 \subseteq \mathcal{B}_1$.

Given a topological space (X, Ω), one can consider a subset $Y \subseteq X$; the set $\Omega_Y = \{O \cap Y : O \in \Omega\}$ is a topology on Y, called the *subspace topology* induced by Ω on Y.

A function $f : (X, \Omega) \to (X', \Omega')$ is called *open* (closed) iff the image of each open (closed, resp.) set is still an open (closed, resp.) set. It is (topologically) *continuous* iff the counter-image of each open set is an open set (i.e., for all $O' \in \Omega'$ one has $f^{-1}(O') \in \Omega$). In particular, a bijection which is continuous and has a continuous inverse is called a *homeomorphism*.

The following Lemma is often used when proving that a certain function is continuous:

Lemma A.3.1 (Gaal [Gaa64], Exercise IV.3.1) *Let $f : X \to X'$ be a function between topological spaces, and $\mathcal{S}$ be a subbase for the topology Ω'. If $f^{-1}(S)$ is open for all $S \in \mathcal{S}$, then f is continuous.* □

As a final point, we recall the definition of metric space. A *pseudometric* on a set X is a function $d : X \times X \to \mathbb{R}^+$ (the set of non-negative reals) such that

1. for all $x \in X$, $d(x, x) = 0$;
2. for all $x, y \in X$, $d(x, y) = d(y, x)$;
3. for all $x, y, z \in X$, $d(x, y) + d(y, z) \geq d(x, z)$ (the *triangle inequality*).

We say that d is a *metric* if moreover $d(x, y) = 0$ always implies $x = y$. Finally, d is an *ultrametric* if it is a metric and satisfies a stronger version of the triangle inequality, i.e.,

$$\forall x, y, z \in X.\ \max\{d(x, y), d(y, z)\} \geq d(x, z).$$

A *(pseudo,ultra)metric space* is a set X endowed with a (pseudo,ultra)metric d. Given a pseudometric space (X, d), for every fixed $x \in X$ and every $\epsilon > 0$, one can consider the *open ball of center x and radius ϵ* defined by taking

$$\mathcal{B}_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}.$$

The set $\mathcal{B} = \{\mathcal{B}_\epsilon(x) : x \in X, \epsilon > 0\}$ is a subbase for a topology Ω_d on X, which is called the *topology induced by the metric d*.

In particular, it is easy to see that

- a set $A \subseteq X$ is open iff, for every $x \in A$, there exists some $\epsilon_x > 0$ such that $\mathcal{B}_{\epsilon_x}(x) \subseteq A$;
- a function $f : (X, d) \to (X', d')$ between pseudometric spaces is topologically continuous if, for every $x \in X$ and every $\epsilon > 0$, there exists a $\delta_{x,\epsilon} > 0$ such that $f(\mathcal{B}_{\delta_{x,\epsilon}}(x)) \subseteq \mathcal{B}_\epsilon(f(x))$, which is the standard version of continuity used when dealing with real numbers.

An alternative (equivalent) way to define the Euclidean topology on $\mathbb{R}$ (and, in general, on its powers) is the following; for each $n \in \omega$, define $d_n : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$ as follows:

$$d_n(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

The function d_n is a metric, and the topology induced on $\mathbb{R}^n$ by d_n is the standard Euclidean topology; in particular, the topology of $\mathbb{R}$ induced by the function

$$d_1(x, y) = |x - y|$$

is the same as the one we have defined before by taking as subbase the set of (half-unbounded) open intervals.

Bibliography

[Abr87] S. Abramsky. Domain theory and the logic of observable properties. Ph. D. thesis, Queen Mary College, Univ. of London, 1987.

[Arm39] W.E. Armstrong. The determinateness of the utility function. *Economics Journal*, 49:453–467, 1939.

[BB95] S. Basagni and P. Boldi. Causal equivalence and congruence between stable event structures. Technical report, no. RI-DSI-147-95, Università di Milano, Dipartimento di Scienze dell'Informazione, Milano, 1995.

[BCS93] P. Boldi, F. Cardone, and N. Sabadini. Concurrent automata, prime event structures and universal domains. In M. Droste and Y. Gurevich, editors, *Semantics of Programming Languages and Model Theory*. Gordon & Breach Science Publications, 1993.

[Bed88] M.A. Bednarczyk. Categories of asynchronous systems. Ph.D. thesis, University of Sussex, Department of Computer Science, 1988.

[Ben91] J. Van Benthem. *The logic of time — A model-theoretic investigation into the varieties of temporal ontology and temporal discourse*. Synthese Library, Vol. 156. Kluwer Academic Publishers, 1991. Second Edition.

[Ber78] G. Berry. Stable models of typed λ-calculi. In *Proc.* 5^{th} *Coll. on Automata, Languages and Programming, LNCS 62*, pages 72–89. Springer-Verlag, 1978.

[Ber79] G. Berry. Modèles complètement adéquats et stables des lambda-calculs typés. Thèse de Doctorat d'Etat, Université de Paris VII, Paris, 1979.

[BG92] S. Brookes and S. Geva. Stable and sequential functions on scott domains. Technical Report CMU-CS-92-121, School of Computer Science, Carnegie Mellon University, 1992.

[BM76] J.L. Bell and M. Machover. *A course in mathematical logic*. North-Holland, 1976.

[Bog82] K.P. Bogart. Some social science applications of ordered sets. In I. Rival, editor, *Ordered Sets, Proceedings of the NATO Advanced Study Institute*, pages 759–790. D. Reidel Publishing Company, 1982.

[Bol95] P. Boldi. Computability and complexity over the reals. NeuroCOLT project, ESPRIT Technical Report no. NC–TR–95–015, Royal Holloway University, London, 1995.

[Bol96] P. Boldi. Maximal chains and antichains in strongly noetherian semiorders. *Fundamenta Informaticae*, 1(27):59–80, 1996.

[Bor46] E. Borel. *Les Paradoxes de l'Infini.* Gallimard, 1946.

[Cam90] P. Cameron. *Oligomorphic Permutation Groups.* Cambridge University Press, 1990.

[Car58] R. Carnap. *Introduction to Symbolic Logic and its Applications.* Dover books on science. New York Dover Publications, Inc., 1958.

[Cau40] A. Cauchy. Sur les moyens d'éviter les erreurs dans les calculs numériques. *Comptes Rendus*, 11:789–798, 1840.

[Coq91] T. Coquand. Constructive topology and combinatorics. In *Constructivity in computer science, LNCS 613*, number 613 in Lecture Notes in Computer Science, pages 159–164. Springer-Verlag, Berlin-Heidelberg-New York, 1991.

[Coq92] T. Coquand. A constructive proof of Tychonoff's theorem. *Journal of Symbolic Logic*, 57:28–32, 1992.

[Cur86] P.-L. Curien. *Categorical Combinators, Sequential Algorithms and Functional Programming.* Pitman, 1986.

[DG93] M. Droste and R. Göbel. Universal domains and the amalgamation property. *Mathematical Structures in Computer Science*, 3:137–159, 1993.

[Die90] V. Diekert. *Combinatorics on Traces, LNCS 454.* Springer-Verlag, 1990.

[Doo94] J.L. Doob. *Measure Theory.* Graduate Texts in Mathematics. Sprinver-Verlag, 1994.

[DR95] V. Diekert and G. Rozenberg, editors. *The Book of Traces.* World Scientific, Singapore, 1995.

[Dro89] M. Droste. Event structures and domains. *Theoretical Computer Science*, 68:37–47, 1989.

[Dro90] M. Droste. Concurrency, automata and domains. In *17th International Colloquium on Automata, Languages and Programming , LNCS 443*, pages 195–208. Springer-Verlag, 1990.

[Dro91] M. Droste. Universal homogeneous event structures and domains. *Information and Computation*, 94:48–61, 1991.

[Dum75] M. Dummett. Wang's paradox. *Synthese*, 30:301–324, 1975. Reprinted in *Truth and Other Enigmas*, Duckworth, London, 1978, pp. 248–268.

[ER63] P. Erdős and A. Rényi. Asymmetric graphs. *Acta Mathematica Academiae Scientiarum Hungaricae*, 14:295–315, 1963.

[ES74] P. Erdős and J. Spencer. *Probabilistic methods in combinatorics.* Probability and Mathematical Statistics. Academic Press, 1974.

[EZZ86] M.H. El-Zahar and N. Zaguia. Antichains and cutsets. In *Combinatorics and Ordered Sets*, Contemporary Mathematics 57, pages 227–261, 1986.

[Fis85] P.C. Fishburn. *Interval orders and interval graphs.* Series in Discrete Mathematics. Wiley-Interscience Publications, 1985.

[Gaa64] S.A. Gaal. *Point Set Topology.* Pure and Applied Mathematics. Academic Press, 1964.

[GH62] A. Ghouila-Houri. Caractérisation des graphes non orientés dont on peut orienter les arêtes de manière à obtenir le graphe d'une relation d'ordre. *C.R. Académie des Sciences de Paris*, 254:1370–1371, 1962.

[Gia93] P. Di Gianantonio. A functional approach to computability on real numbers. Ph.D. thesis (TD–6/93), Università di Pisa, Dipartimento di Informatica, Pisa, 1993.

[Gir86] J.-Y. Girard. The system F of variable types, fifteen years later. *Theoretical Computer Science*, 45:159–192, 1986.

[Gir87] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50:1–43, 1987.

[GLT89] J.-Y. Girard, Y. Lafont, and P. Taylor. *Proofs and Types.* Cambridge University Press, Cambridge, 1989.

[Gol80] M.C. Golumbic. *Algorithmic graph theory and perfect graphs.* Computer Science and Applied Mathematics. Academic Press, 1980.

[Goo77] N. Goodman. *The structure of appearance.* D. Reidel Publishing Company, Dordrecht, third edition, 1977.

[Gri69] G.A. Grillet. Maximal chains and antichains. *Fundamenta Mathematicae*, 65:157–167, 1969.

[GS90] C.A. Gunter and D.S. Scott. Semantic domains. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science, Volume B*, pages 633–674, Amsterdam, 1990. North Holland.

[Gun87] C. Gunter. Universal profinite domains. *Information and Computation*, 72:1–30, 1987.

[H$^+$68] A.W. Holt et al. Information system theory project. Technical report RADC–TR–68–305, Applied Data Research, Inc., 1968.

[Hol74a] A.W. Holt. Final Report for the project "Development of theoretical foundations for description and analysis of discrete information systems". Technical Report CADD–7405–2011, Massachusetts Computer Associates, Wakefield, Mass., 1974.

[Hol74b] A.W. Holt. First semi-annual Report for the project "Development of theoretical foundations for description and analysis of discrete information systems". Technical Report CADD–7406–0611, Massachusetts Computer Associates, Wakefield, Mass., 1974.

[Hol80] A.W. Holt. A mathematical model of continuous discrete behavior. Technical report, Massachusetts Computer Associates, Wakefield, Mass., 1980.

[Kal67] L. Kalmar. Value, synonymy, and translation. In *Symposium Mashperevod '67, Budapest*, 1967.

[Kat94] R.H. Katz. *Contemporary Logic Design.* The Benjamin/Cummings Publishing Company, Inc., 1994.

[Kel55] J.L. Kelley. *General Topology.* American Book Company, New York, 1955.

[KKM90] E. Khalimsky, R. Kopperman, and P.R. Meyer. Computer graphics and connected topologies on finite ordered sets. *Topology and its Applications*, 36:1–17, 1990.

[KP78] G. Kahn and G. D. Plotkin. Domaines concrèts. Rapport 336, IRIA Laboria, 1978.

[Lam78] L. Lamport. Time, clocks, and the ordering of events in a distributed system. *ACM Communications*, 21(7):558–565, 1978.

[LM73] B. Leclerc and B. Monjardet. Ordres C.A.C. *Fundamenta Mathematicae*, 79:11–22, 1973.

[Luc56] R.D. Luce. Semiorders and a theory of utility discrimination. *Econometrica*, 24:178–191, 1956.

[Mac71] S. MacLane. *Categories for the Working Mathematician.* Springer-Verlag, 1971.

[Maz77] A. Mazurkiewicz. Concurrent program schemes and their interpretations. Technical Report DAIMI PB-78, Aarhus University, Aarhus, 1977.

[Men40] K. Menger. Topology without points. *Rice Institute Pamphlets*, 27:80–107, 1940.

[NPW81] M. Nielsen, G.D. Plotkin, and G. Winskel. Petri nets, event structures and domains, Part I. *Theoretical Computer Science*, 13:85–108, 1981.

[Par83] R. Parikh. The problem of vague predicates. In R.S.Cohen and M.W.Wartofsky, editors, *Language, logic and method*, pages 241–261. D.Reidel Publishing Company, 1983.

[Pat70] S. Patil. Coordination of asynchronous events. PhD Thesis 72, MIT, Project MAC, 1970.

[Pet62] C.A. Petri. *Kommunikation mit Automaten.* PhD thesis, Universität Darmstadt, 1962. translated by Clifford F. Greene, Jr. as a Supplement to Technical Documentary Report #1, prepared for the Rome Air Development Center Contract #AF30(602)–3324, Summer 1965.

[Pet77] C.A. Petri. Non-sequential processes. Interner Bericht ISF-77-5, Gesellschaft für Mathematik und Datenverarbeitung, Bonn, 1977.

[Pet79] C.A. Petri. Concurrency as a basis of system thinking. In F.V. Jensen, B.H. Mayoh, and K.K. Moller, editors, *Proc. of 5th Scandinavian Logic Symposium.* Aalborg University Press, 1979.

[Pet80a] C.A. Petri. Concurrency. In *Net theory and Applications: Proceedings of the Advanced Course on General Net Theory of Processes and Systems, LNCS 84*, pages 251–260. Springer-Verlag, 1980.

[Pet80b] C.A. Petri. General net theory. In *Net theory and Applications: Proceedings of the Advanced Course on General Net Theory of Processes and Systems, LNCS 84*, pages 1–19. Springer-Verlag, 1980.

[Pet82a] C.A. Petri. Fundamentals of the representation of discrete processes. ISF–Report 82.04, Gesellschaft für Mathematik und Datenverarbeitung, Bonn, 1982. Translation, by H.J. Genrich and P.S. Thiagarajan of: *Grundsätzliches zur Beschreibung diskreter Prozesse*, in *3. Colloquium über Automatentheorie*, Birkhäuser Verlag, Basel, 1967.

[Pet82b] C.A. Petri. State transition structures in physics and computation. *International Journal of Theoretical Physics*, 21(10/11), 1982.

[Pet87] C.A. Petri. Concurrency theory. In W. Brauer, W. Reisig, and G. Rozenberg, editors, *Advances in Petri Nets, LNCS 254*, pages 4–24. Springer-Verlag, 1987.

[Pet96] C.A. Petri. Nets, time and space. *Theoretical Computer Science*, 153:3–41, 1996.

[Plo78] G.D. Plotkin. $\mathbf{T}^\omega$ as a universal domain. *Journal of Computer and Systems Sciences*, 17:209–236, 1978.

[Poi03] H. Poincaré. *La Science et l'hypothèse.* Flammarion, Paris, 1903. English translation as *Science and Hypothesis*, Dover, New York, 1952.

[Pos71] T. Poston. *Fuzzy Geometry.* PhD thesis, University of Warwick, 1971.

[PS87] C.A. Petri and E. Smith. Concurrency and continuity. In *Advances in Petri Nets 1987, LNCS 266*, pages 273–292. Springer-Verlag, 1987.

[Rad67] R. Rado. Universal graphs. In F. Harary, editor, *A Seminar on Graph Theory*, pages 83–85, New York, 1967. Holt, Rinehart and Winston.

[Riv86] I. Rival. Stories about order and the letter N. In *Combinatorics and ordered sets*, Contemporary Mathematics 57, pages 263–285, 1986.

[Rob79] F.S. Roberts. *Measurement theory.* Encyclopedia of Mathematics and its Applications, 7. Addison-Wesley, 1979.

[RT91] B. Rozoy and P.S. Thiagarajan. Event structures and trace monoids. *Theoretical Computer Science*, 91:285–313, 1991.

[Rus37] B. Russell. *The Principles of Mathematics.* George Allen & Unwin Ltd., London, 1937.

[Sco70] D. Scott. Lattice theory, data types and semantics. In P. Rustin, editor, *Formal Semantics of Programming Languages*, pages 65–106, Englewood Cliffs, NJ, 1970. Prentice-Hall.

[Sco76] D. Scott. Data types as lattices. *SIAM Journal on Computing*, 5:522–587, 1976.

[Sco82] D. Scott. Domains for denotational semantics. In *Proc. 9th International Colloquium on Automata, Languages and Programming, LNCS 140*, pages 577–613. Springer-Verlag, 1982.

[Sei80] C.L. Seitz. System timing. In C. Mead and L. Conway, editors, *Introduction to VLSI systems*, pages 218–262. Addison-Wesley, 1980.

[Shi82] M.W. Shields. Non-sequential behaviour: I. Technical Report CSR-120-82, University of Edinburgh, 1982.

[Shi85] M.W. Shields. Concurrent machines. *Computer Journal*, 28:449–465, 1985.

[Shr68a] Yu.A. Shreyder. A mathematical model of the theory of classification. *Nauchno-tekhnicheskaya informatsiya, Series 2*, 10:7–14, 1968.

[Shr68b] Yu.A. Shreyder. On the question of the structure of scientific collectives. In *Questions in the theory and practice of informational and information-search systems.* Ukr. NIINTI, 1968.

[Shr71] Yu.A. Shreyder. Tolerance spaces. *Journal of Cybernetics*, 1(2):115–122, 1971.

[Smi93] E. Smith. Comparability orders and measurement. In *Advances in Petri Nets 1993, LNCS 674*, pages 371–405. Springer-Verlag, 1993.

[Smy92] M.B. Smyth. Topology. In S. Abramsky, D. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science Volume 1*, pages 641–761. Oxford University Press, 1992.

[Smy95] M.B. Smyth. Semi-metrics, closure spaces and digital topology. *Theoretical computer science*, 151:257–276, 1995.

[SS58] D. Scott and P. Suppes. Foundational aspects of theories of measurement. *Journal of Symbolic Logic*, 23:113–128, 1958.

[Sta89a] E.W. Stark. Compositional relational semantics for indeterminate dataflow networks. In *Category Theory and Computer Science, LNCS 389*, pages 52–74. Springer-Verlag, 1989.

[Sta89b] E.W. Stark. Connections between a concrete and an abstract model of concurrent systems. In *Mathematical Foundations of Programming Semantics, LNCS 442*, pages 53–79. Springer-Verlag, 1989.

[Sün95] P. Sünderhauf. A faithful computational model of the real numbers. *Theoretical Computer Science*, 151:277–294, 1995.

[Thi96a] H. Thiele. On isomorphisms between the De Morgan algebra of fuzzy tolerance relations and De Morgan algebras of fuzzy clusterings. In *Proceedings of the 5th IEEE International Conference on Fuzzy Systems*, 1996. To appear.

[Thi96b] H. Thiele. On mathematical foundations of fuzzy cluster analysis. *Multiple-Valued Logic and Soft Computing*, 1996. To appear.

[TS95] H. Thiele and N. Schmechel. On the mutual definability of fuzzy equivalence relations and fuzzy partitions. In *Proceedings of the 4th IEEE International Conference on Fuzzy Systems*, volume 3, pages 1383–1390, 1995.

[Wey49] H. Weyl. *Philosophy of Mathematics and Natural Science*. Princeton University Press, 1949. Revised English edition.

[Wie14] N. Wiener. Contributions to the theory of relative position. *Proceedings of the Cambridge Philosophical Society*, 17:441–449, 1912–1914.

[Wie16] N. Wiener. A new theory of measurement: A study in the logic of mathematics. *Proceedings of the London Mathematical Society*, 19:181–205, 1914–1916.

[Win80] G. Winskel. Events in computation. Ph.D. thesis, University of Edinburgh, Department of Computer Science, Edinburgh, 1980.

[Win87] G. Winskel. Event structures. In *Petri Nets: Applications and Relationships to other models of concurrency, LNCS 255*, pages 325–392. Springer-Verlag, 1987.

[Win91] G. Winskel. Categories of models for concurrency. Notes, Advanced school on the algebraic, logical, and categorical foundations of concurrency, Gargnano del Garda, 1991.

[Wri75] C. Wright. On the coherence of vague predicates. *Synthese*, 30:325–365, 1975.

[ZB70] E. Zeeman and O. Buneman. Tolerance spaces and the brain. In *On the Path to Theoretical Biology*, Moscow, 1970.

[Zee62] E.C. Zeeman. The topology of the brain and visual perception. In M.K. Fort, editor, *The topology of 3-Manifolds*, pages 240–256. Prentice Hall, Englewood Cliffs / NJ, 1962.